Università degli Studi di Milano-Bicocca - Laurea Magistrale in Matematica

Esame di metodi matematici per l'analisi economica – controllo ottimo

27 Gennaio 2020

Cognome:____________________________ nome:____________________________

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1. (6 punti) Using the variational approach solve the problem

$$\begin{cases} \min_u T \\ \ddot{x} = u \\ x(0) = \dot{x}(0) = -1 \\ x(T) = \dot{x}(T) = 0 \\ |u| \leq 1 \end{cases}$$

2. (6 punti) Using the Dynamic Programming solve the problem

$$\begin{cases} \min_u \int_0^2 (x - u) \, \mathrm{d}t + x(2) \\ \dot{x} = 1 + u^2 \\ x(0) = 1 \end{cases}$$

In order to solve BHJ equation, we suggest to find the solution in the family of functions

$\mathcal{F} = \{V(t, x) = A + Bt + Ct^2 + D \ln(3 - t) + E(3 - t)x, \text{ with } A, B, C, D, E \text{ constants}\}.$

3. (6 punti) Let us consider the following optimal control problem

$$\begin{cases} J(\mathbf{u}) = \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u}) \, \mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}) \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to U \subset \mathbb{R}^k, \ \mathbf{u} \text{ admissible}\} \end{cases} \tag{1}$$

with $\boldsymbol{\alpha}$, t_0 and t_1 fixed.

i. With the appropriate and minimal assumptions, give the necessary condition of Pontryagin for the problem (1);

ii. with the appropriate assumptions, give the sufficient condition of Mangasarian for the problem (1); it is also required the proof.

4. (6 punti) Let us consider the "moonlanding problem":

$$\begin{cases} \max_{u \in \mathcal{C}} m(T) \\ \dot{h} = v \\ \dot{v} = \frac{u}{m} - g \\ \dot{m} = -ku \\ h(0) = h_0, \qquad h(T) = 0 \\ v(0) = v_0, \qquad v(T) = 0 \\ m(0) = M + F \\ m(t) \geq 0, \qquad h(t) \geq 0 \\ \mathcal{C} = \{u : [0, T] \to [0, \alpha], \ \text{admissible}\} \end{cases}$$

where $h_0,\ M,\ F,\ g,\ -v_0,\ k$ and α are positive and fixed constants, and the final time T is free.

i. Introduce the problem with all the details;

ii. prove that the optimal solution has at most one switching point.

5. (6 punti) Let us consider the following zero sum differential game

$$\begin{cases} \text{Player I: } \max\limits_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2), \qquad \text{Player II: } \min\limits_{\mathbf{u}_2} J(\mathbf{u}_1,\mathbf{u}_2) \\ J(\mathbf{u}_1,\mathbf{u}_2) = \displaystyle\int_0^T f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,dt + \psi(\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \end{cases} \tag{2}$$

where T is positive and fixed, U_i are the control sets for the players.

i. Introduce, with all the details, the definitions of lower value function V^- and upper value function V^+ (in particular provide the definitions of control and nonanticipative strategy). When the problem (2) admits value function V ?

ii. Let us consider the game

$$\begin{cases} \text{Player I: } \max\limits_{u_1} J(u_1,u_2) \qquad \text{Player II: } \min\limits_{u_2} J(u_1,u_2) \\ |u_1| \le 1 \qquad\qquad |u_2| \le 1 \\ J(u_1,u_2) = \displaystyle\int_0^\infty \operatorname{sgn}(x)\left(1 - e^{-|x|}\right) e^{-t}\,dt \\ \dot{x} = (u_1 - u_2)^2 \\ x(0) = x_0 \end{cases} \tag{3}$$

Prove that it does not admit value function.

iii. Now consider the problem

$$\begin{cases} \text{Player I: } \max\limits_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2), \qquad \text{Player II: } \min\limits_{\mathbf{u}_2} J(\mathbf{u}_1,\mathbf{u}_2) \\ J(\mathbf{u}_1,\mathbf{u}_2) = \psi(T,\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ (T,\mathbf{x}(T)) \in \partial S \end{cases} \tag{4}$$

with T free and the closed target set $S \subset [0,\infty) \times \mathbb{R}^n$. Let us assume that the problem admits a value function in C^1 and that the Isaacs condition is satisfied (if is the case, add further necessary assumptions). Give a geometric proof of the Isaacs equation for the problem (4).