T test for countinous outcomes: Testing a claim about a mean

Example

The authors obtained times of sleep for randomly selected adult subjects included in the National Health and Nutrition Examination Study, and those times (hours) are : 4 8 4 4 8 6 9 7 7 10 7 8

A common recommendation is that adults should sleep between 7 hours and 9 hours each night.

Use the statistical test with a 0.05 significance level to test the claim that the mean amount of sleep for adults is less than 7 hours.

n = 12
$\bar{x}$ = 6.83333333 hours
s = 1.99240984 hours

$$H_0\colon\ \mu = 7$$
$$H_1\colon\ \mu < 7$$

Test a claim about a (population) mean μ

By the central limit theorem we have seen that

For all samples of the same size n (with n>30), the sampling distribution of the mean of a random variable with mean μ and standard deviation σ can be approximated by a normal distribution with mean μ and standard deviation σ/√n.

Thus:

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

However σ is not usually known, so we have to estimate it by:

$$s = \hat{\sigma} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}}$$

Test a claim about a (population) mean μ

When σ is estimated the standardised difference:

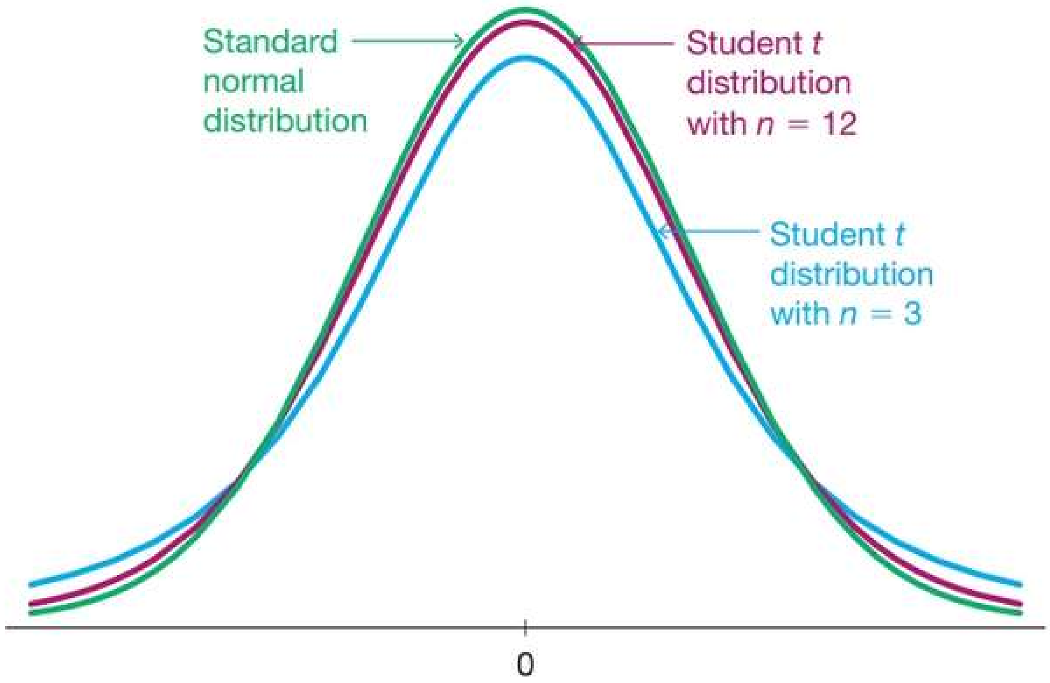
$$\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

becomes $$\frac{\bar{x} - \mu}{s/\sqrt{n}} \sim Tstudent_{df=n-1}$$

df: number of degrees of freedom (number of sample values that can vary after certain restrictions have been imposed on all data values)
df=n-1

For large sample size t-student ~N(0,1)

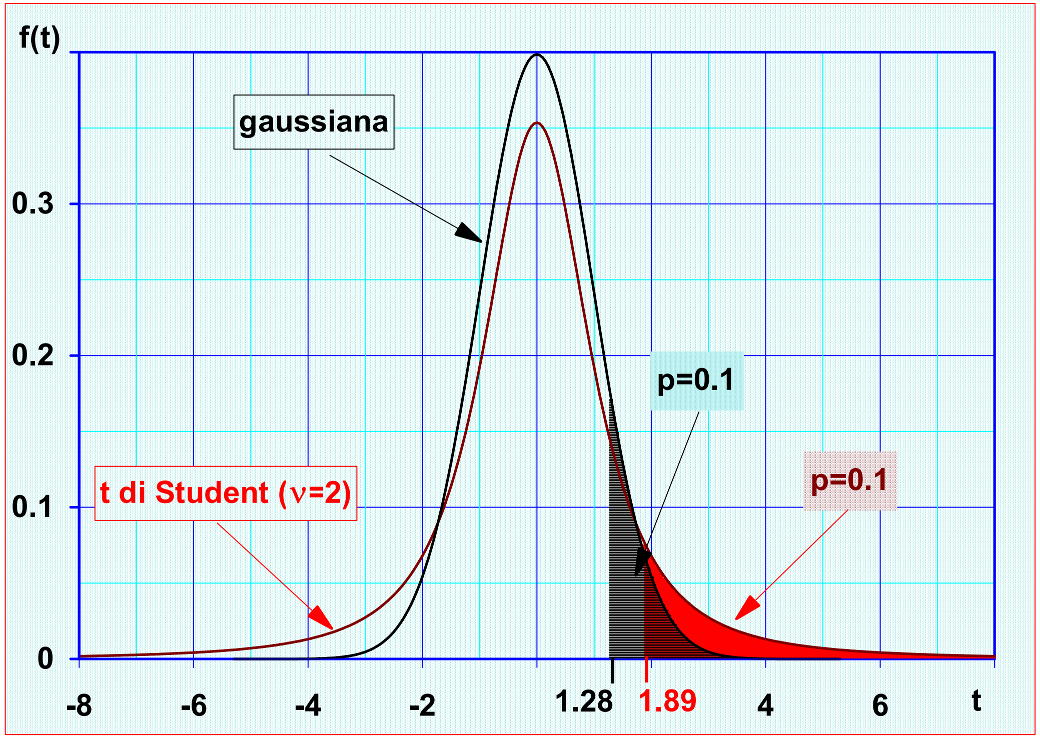
Student t distribution - William Gosset (1876–1937)



Student t distribution - William Gosset (1876–1937)

- The Student t distribution has the same general symmetric bell shape as the standard normal distribution, but has more variability (with wider distributions), as we expect with small samples.
- The Student t distribution has a mean of $t = 0$ (just as the standard normal distribution has a mean of $z = 0$)
- The standard deviation of the Student t distribution varies with the sample size, but it is greater than 1 (unlike the standard normal distribution, which has $s = 1$).
- The Student t distribution is different for different sample sizes.
- As the sample size n gets larger, the Student t distribution gets closer to the standard normal distribution.

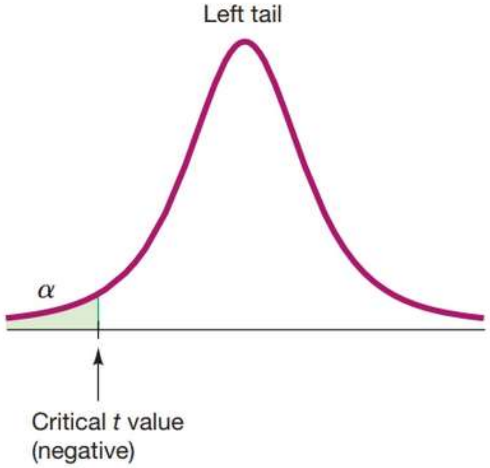
Student t distribution - William Gosset (1876–1937)



Student's t distribution has percentiles with an absolute value that is higher than that of the corresponding Gaussian percentiles, the lesser the number of degrees of freedom.

For example, the 90th percentile of the Gaussian standard is 1.282, while the corresponding percentiles of Student's with 1, 2, 3 and 9 g.d.l. are respectively 3.078, 1.886, 1.638 and 1.383.

Student t table



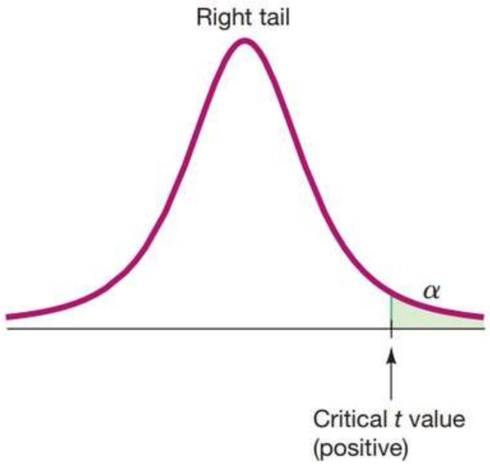


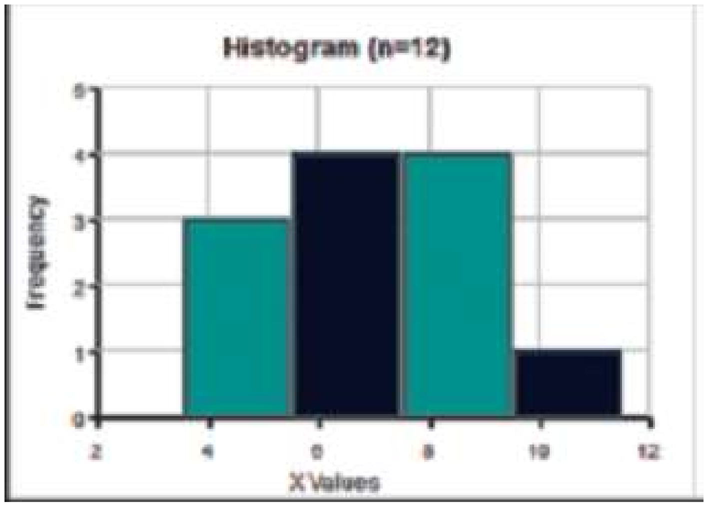
TABLE A-3 *t* Distribution: Critical *t* Values

	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
Degrees of Freedom	**Area in Two Tails**				
	0.01	0.02	0.05	0.10	0.20
1	63.657	31.821	12.706	6.314	3.078
2	9.925	6.965	4.303	2.920	1.886
3	5.841	4.541	3.182	2.353	1.638
4	4.604	3.747	2.776	2.132	1.533
5	4.032	3.365	2.571	2.015	1.476
6	3.707	3.143	2.447	1.943	1.440
7	3.499	2.998	2.365	1.895	1.415
8	3.355	2.896	2.306	1.860	1.397
9	3.250	2.821	2.262	1.833	1.383
10	3.169	2.764	2.228	1.812	1.372
11	3.106	2.718	2.201	1.796	1.363
12	3.055	2.681	2.179	1.782	1.356
13	3.012	2.650	2.160	1.771	1.350
14	2.977	2.624	2.145	1.761	1.345
15	2.947	2.602	2.131	1.753	1.341
16	2.921	2.583	2.120	1.746	1.337
17	2.898	2.567	2.110	1.740	1.333
18	2.878	2.552	2.101	1.734	1.330
19	2.861	2.539	2.093	1.729	1.328
20	2.845	2.528	2.086	1.725	1.325
21	2.831	2.518	2.080	1.721	1.323
22	2.819	2.508	2.074	1.717	1.321
23	2.807	2.500	2.069	1.714	1.319
24	2.797	2.492	2.064	1.711	1.318
25	2.787	2.485	2.060	1.708	1.316
26	2.779	2.479	2.056	1.706	1.315
27	2.771	2.473	2.052	1.703	1.314
28	2.763	2.467	2.048	1.701	1.313
29	2.756	2.462	2.045	1.699	1.311

T-test: requirements

(1) The sample is a simple random sample. OK
(2) The population is normally distributed or n > 30. ?

The sample size is n = 12, which does not exceed 30, so we must determine whether the sample data appear to be from a normally distributed population.
The accompanying histogram, along with the apparent absence of outliers, indicate that the sample appears to be from a population with a distribution that is approximately normal.
Both requirements are satisfied



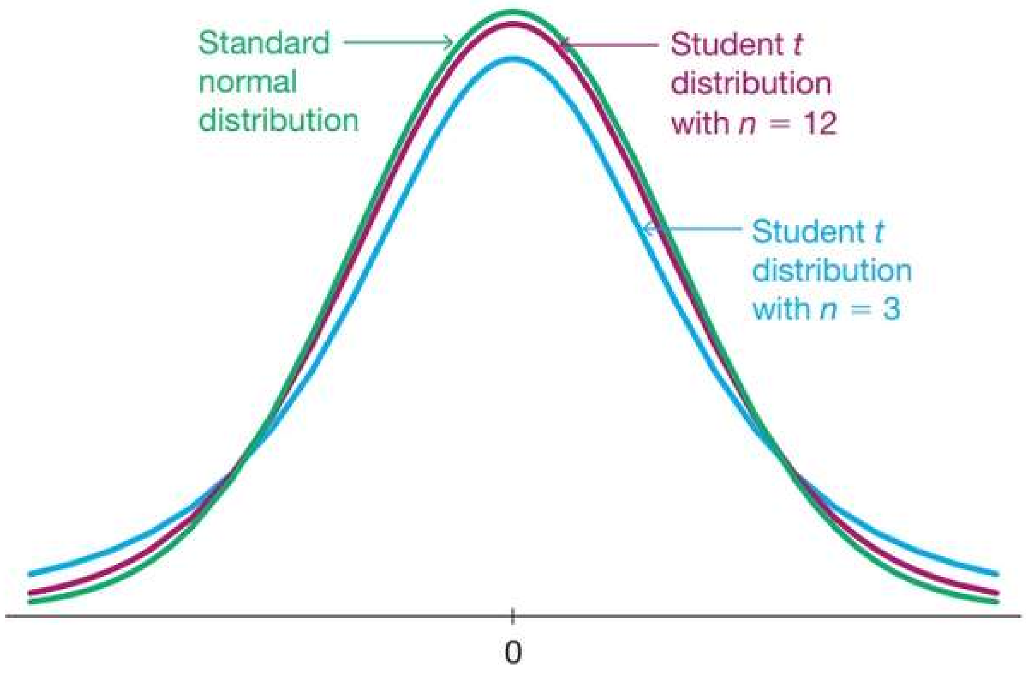
t-test:

As requirements are satisfied and because the claim is made about the population mean μ, the sample statistic most relevant to this test is the sample mean $\bar{x}$, and we use the t distribution:

$$\frac{\bar{x}-\mu_0}{s/\sqrt{n}} \sim Tstudent_{df=n-1} \text{ under } \mathrm{H}_0$$

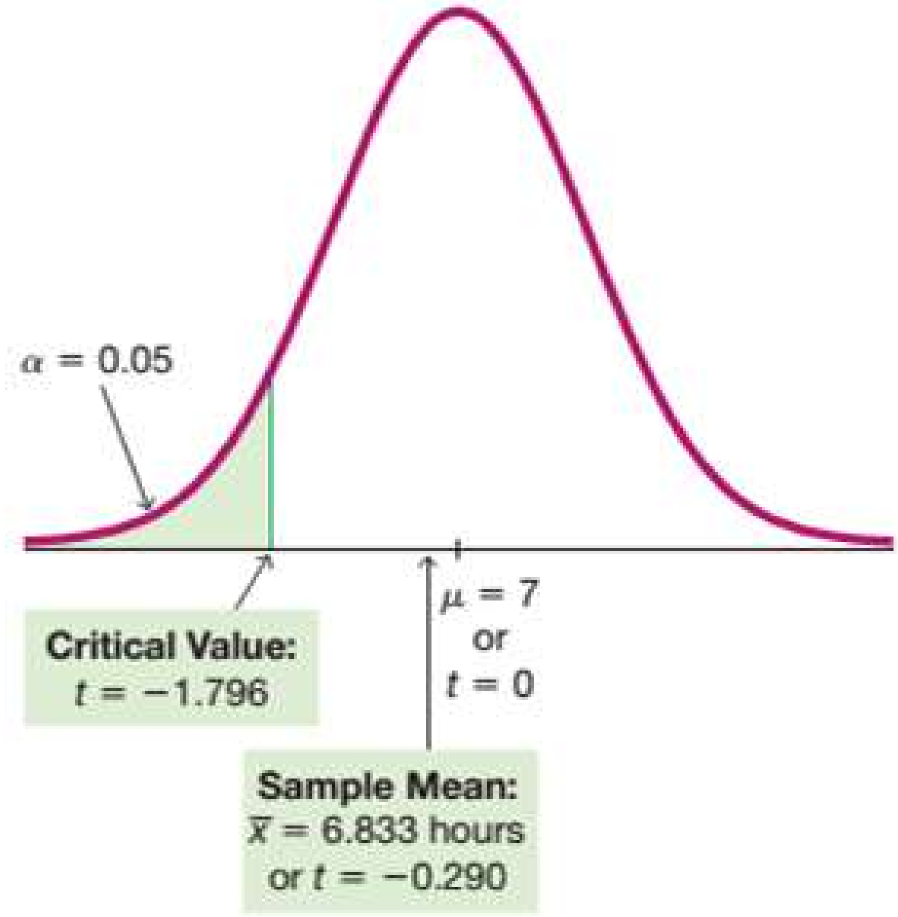
df=n-1

For large sample size t-student ~N(0,1)



T-test:

H_0: $\mu = \mu_0 = 7$
H_1: $\mu < \mu_0 = 7$
α=0.05
df=12-1=11
Critical value of t =- 1.796



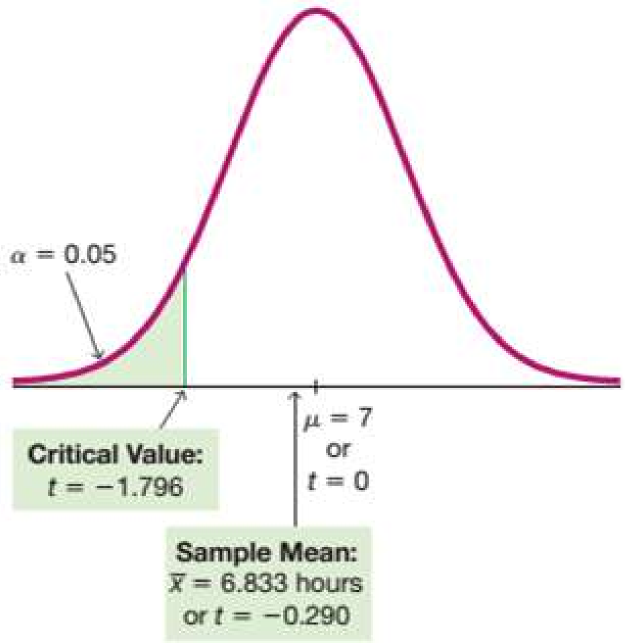
$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} =$$
$$= \frac{6.833 - 7}{1.992/\sqrt{12}} = -0.290$$

Because we fail to reject the null hypothesis, we conclude that there is not sufficient evidence to support the claim that the mean amount of adult sleep is less than 7 hours.

T-test: p-value

H_0: $\mu = 7$
H_1: $\mu < 7$
α=0.05
df=12-1=11

$t = -0.290$



P-Value with Statistical software (i.e. STATA: display t(11,-0.29))

- P-value is 0.3886 (rounded), because it is greater than the significance level of α= 0.05, we fail to reject the null hypothesis.

P-Value without Statistical software:

- Using the test statistic of t = - 0.290 with t-student Table, examine the values of t in the row for df = 11 to see that 0.290 is less than all of the listed t values in the row, which indicates that the area in the left tail below the test statistic of t = - 0.290 is greater than 0.10. In this case, the Table allows us to conclude that the P-value > 0.10, but technology provided the P-value of 0.3886. With a P-value > 0.10, the conclusions are the same as before.

T-test: confidence interval

H_0: $\mu = 7$
H_1: $\mu < 7$
α=0.05
df=12-1=11
Critical value of t = -1.796
Confidence interval at 90% confidence (two tails by definition):

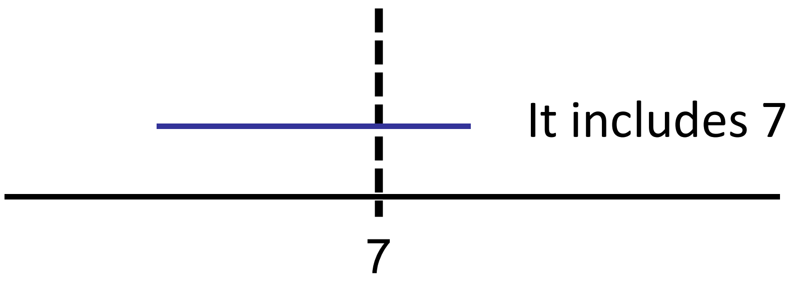
n = 12
$\bar{x}$ = 6.83333333 hours
s = 1.99240984 hours

$$\left[\ \bar{x} - t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \quad ; \quad \bar{x} + t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\ \right]$$

$$\left[6.833 - 1.796 \frac{1.992}{\sqrt{12}} \quad ; \quad 6.833 + 1.796 \frac{1.992}{\sqrt{12}}\right]$$

$$[5.800 \ ; 7.866]$$

With 90% confidence we can say that the mean amount of sleep for adults is between 5.800 and 7.866 hours.



It includes 7: not reject H_0

T-test: relationship between critical value and confidence interval:

H_0: $\mu = \mu_0$
H_1: $\mu \neq \mu_0$

$$\text{if } t_{n-1,\frac{\alpha}{2}} < t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < t_{n-1,\ 1-\frac{\alpha}{2}}$$

not reject H_0

$$\bar{x} - t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} < \mu_0 < \bar{x} + t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

Confidence interval includes μ_0

NOTE: The P-value method and critical value method are equivalent in the sense that they always lead to the same conclusion.

$$t_{n-1,1-\frac{\alpha}{2}} = -t_{n-1,\frac{\alpha}{2}}$$

Test on proportion: relationship between critical value and confidence interval

A confidence interval estimate of a proportion might lead to a conclusion different from that of a hypothesis test.

$$H_0: \pi = \pi_0$$
$$H_1: \pi \neq \pi_0$$

$$z_{\frac{\alpha}{2}} < z = \frac{p-\pi_0}{\sqrt{\pi_0 * (1-\pi_0)/n}} < z_{1-\frac{\alpha}{2}}$$

$$\not\Downarrow$$

$$\mathrm{p} - z_{\frac{\alpha}{2}} \frac{\sqrt{p*(1-p)}}{\sqrt{n}} < \pi_0 < p + z_{\frac{\alpha}{2}} \frac{\sqrt{p*(1-p)}}{\sqrt{n}}$$

Is a confidence interval equivalent to a hypothesis test in the sense that they always lead to the same conclusion?

Proportion	No
Mean	Yes

Exercise

Data Set 2 "Body Temperatures" includes measured body temperatures with these statistics for 12 AM on day 2: n = 106, $\bar{x}$= 36.8 °C (98.20°F), s =0.4 °C (0.62°F). Use a 0.05 significance level to test the common belief that the population mean is 37 °C (98.6°F).

H_0: $\mu = 37$

H_1: $\mu \neq 37$

α=0.05

The critical values are ± 1.984

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{36.8 - 37}{0.4/\sqrt{106}} = -5.15$$

P-value<0.0001

Confidence interval 95%: (36.72;36.88)°C

There is sufficient evidence to warrant rejection of the common belief that the population mean body temperature is 37°C.