Causal Networks

Nonparametric Identification

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In this lecture we will ask and answer the following question "*is it possible to get identifiability without being able to block all backdoor paths*?".

In particular, the lecture presents and discusses the following:

- Frontdoor criterion
- Frontdoor adjustment
- Do-calculus
- Identifiability from the graph

PART I

THE FRONTDOOR ADJUSTMENT

THE BACKDOOR CRITERION

Given an ordered pair of variables (X, Y) in a DAG $\mathcal{G}$, a set of variables $\mathbf{S}$ satisfies the backdoor criterion relative to (X, Y) if no node in $\mathbf{S}$ is a descendant of X, and $\mathbf{S}$ blocks every path between X and Y that contains an arrow into X.

⇒ sufficient for identification

⇒ necessary for identification?

In other words, is it possible to get identifiability without being able to block all backdoor paths?

Consider the century-old debate on the relation between smoking and lung cancer.

In the years preceding 1970, the tobacco industry has managed to prevent antismoking legislation by promoting the theory that the observed correlation between smoking and lung cancer could be explained by some sort of carcinogenic genotype that also induces an inborn craving for nicotine.

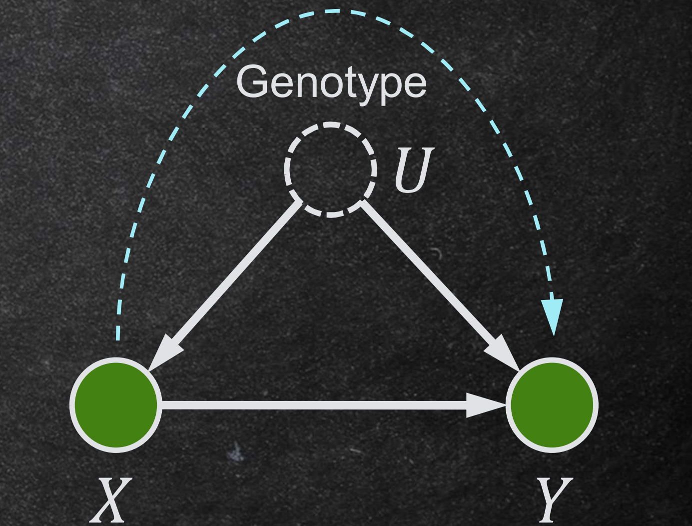

Figure 7.1

The graph (Figure 7.1) does not satisfy the backdoor condition because the variable U is unobserved and hence cannot be used to block the backdoor path $(X \leftarrow U \rightarrow Y)$ from X to Y.

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The causal effect of Smoking (X) on Lung cancer (Y) is not identifiable in this model (Figure 7.1); one can never ascertain which portion of the observed correlation between X and Y is spurious, attributable to their common cause U, $(X \leftarrow U \rightarrow Y)$ and what portion is genuinely causative $(X \rightarrow Y)$.

(We note, however, that even in these circumstances, much compelling work has been done to quantify how strong the (unobserved) associations between both U and X, and U and Y, must be in order to entirely explain the observed association between X and Y)

However, we can go much further by considering the model in Figure 7.2, where an additional measurement is available: amount of Tar deposits (T) in patients' lungs.

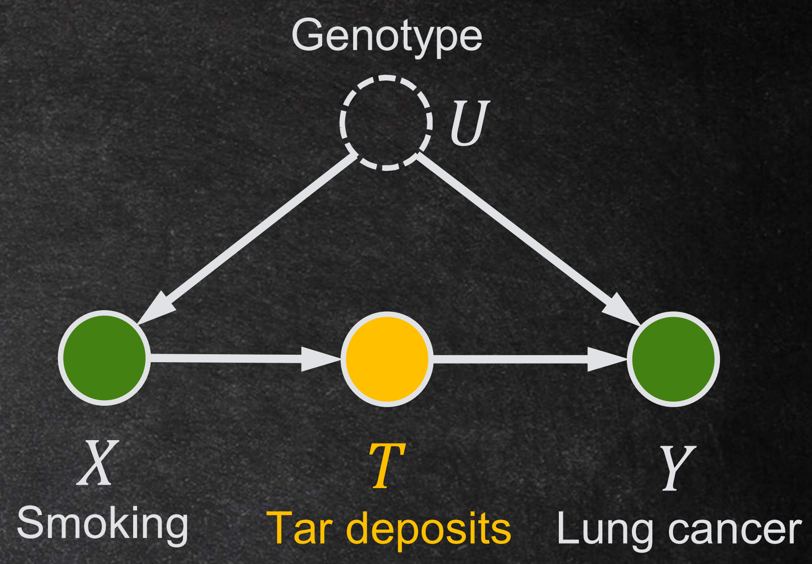

Figure 7.2

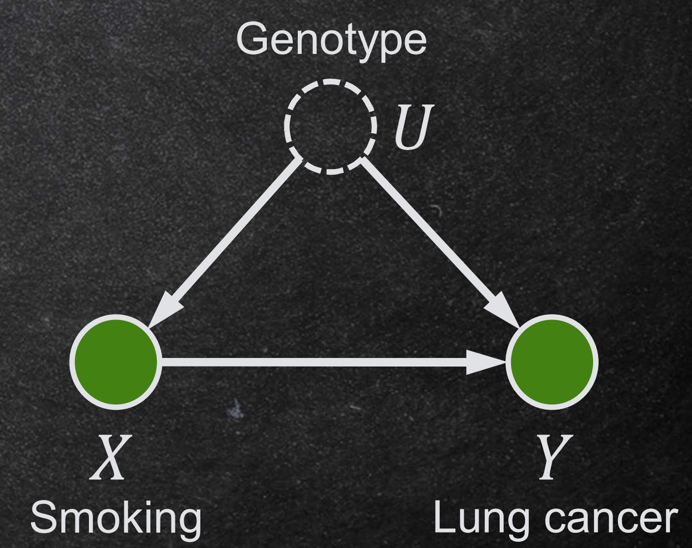

Figure 7.1

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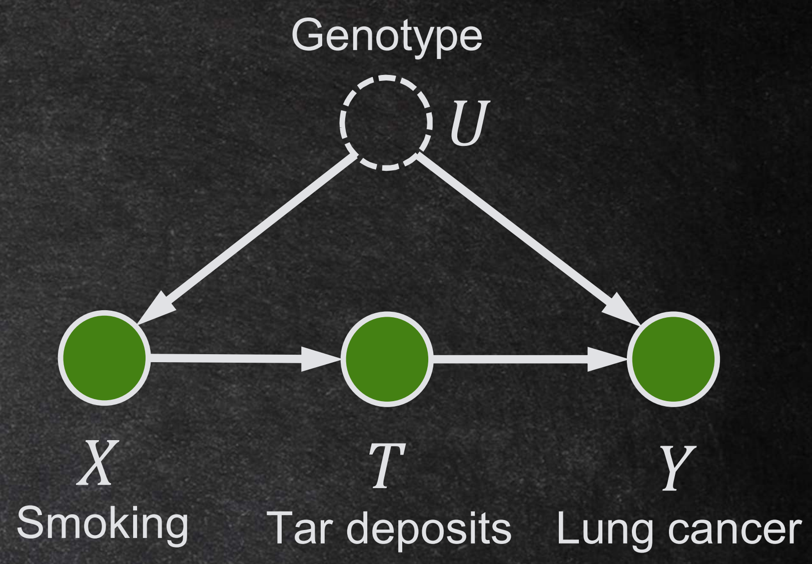


Figure 7.2

The model in Figure 7.2 does not satisfy the backdoor criterion, because there is still no variable capable of blocking the spurious path

$$X \leftarrow U \rightarrow Y.$$

We see, however, that the causal effect of X on Y

$$P\big(Y = y | do(X = x)\big)$$

is nevertheless identifiable in this model, through two consecutive applications of the backdoor criterion.

How can the intermediate variable T help us to assess the effect of X on Y?

The answer is not at all trivial: as the following quantitative example shows, it may lead to heated debate.

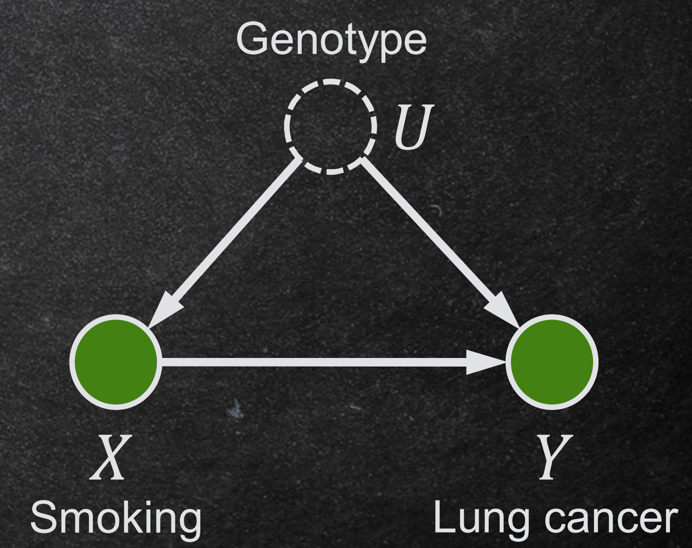


Figure 7.1

Assume that a careful study was undertaken, in which the following factors were measured simultaneously on a randomly selected sample of 800,000 subjects considered to be at very high risk of cancer (because of environmental exposures such as smoking, asbestos, radon, and the like):

- whether the subject smoked
- amount of tar in the subject's lungs
- whether lung cancer has been detected

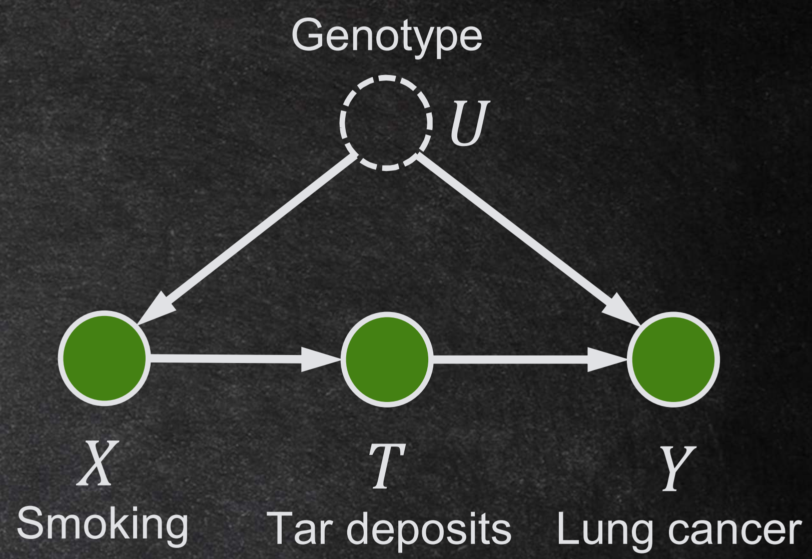


Figure 7.2

Assume that a careful study was undertaken, in which the following factors were measured simultaneously on a randomly selected sample of 800,000 subjects considered to be at very high risk of cancer (because of environmental exposures such as smoking, asbestos, radon, and the like):

- whether the subject smoked (**smokers**, **non smokers**)
- amount of tar in the subject's lungs (**tar**, **no tar**)
- whether lung cancer has been detected (**cancer**, **no cancer**)

The data are presented in the bottom table, where, for simplicity, all three variables are assumed to be binary. All numbers are given in thousands.

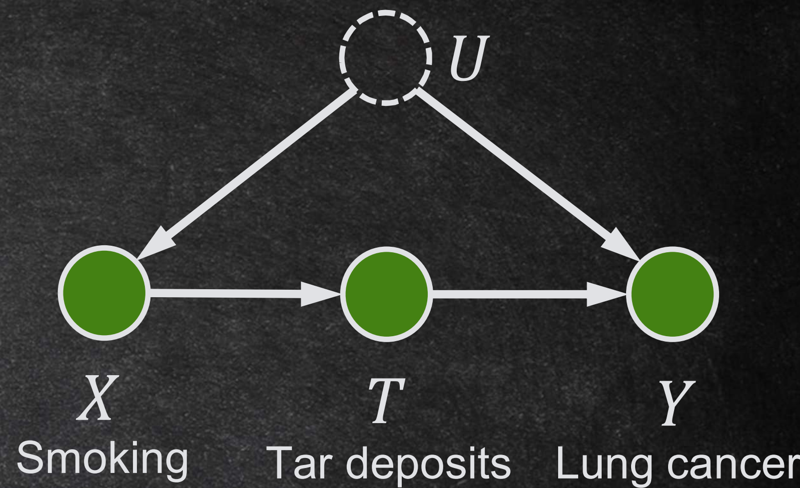
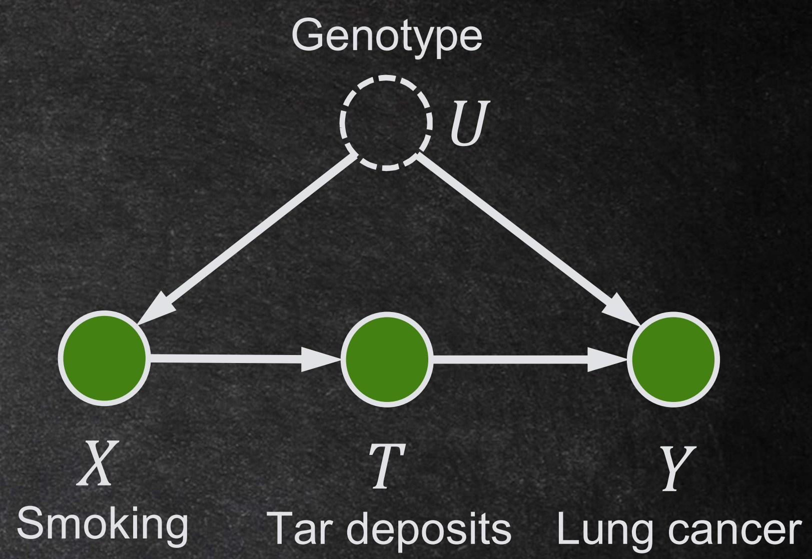


Figure 7.2

	tar		no tar		all subjects	
	400		400		800	
	smokers	non smokers	smokers	non smokers	smokers	non smokers
	380	20	20	380	400	400
no cancer	323 85%	1 5%	18 90%	38 10%	341 85.25%	39 9,75%
cancer	57 15%	19 95%	2 10%	342 90%	59 14.75%	361 90.25%

Two opposing interpretations can be offered for these data.

- The tobacco industry argues that the table proves the beneficial effect of smoking.



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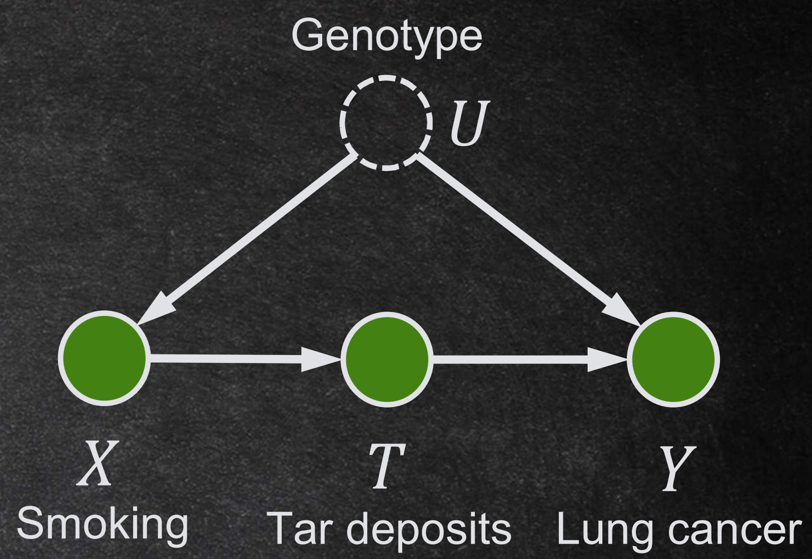
- The tobacco industry argues that the table proves the beneficial effect of smoking.

  They point to the fact that only **14.75%** of the **smokers** have developed lung cancer, compared to **90.25%** of the **non smokers**.



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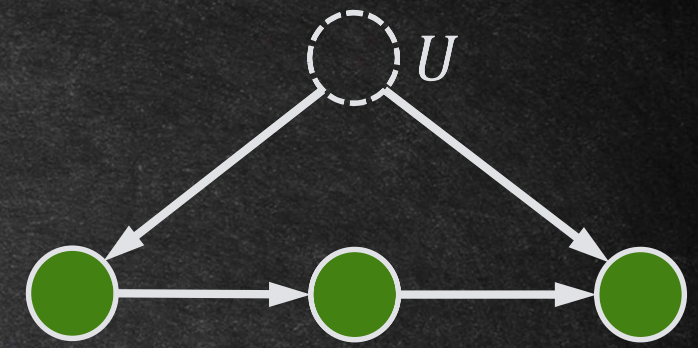
  They point to the fact that only 14.75% of the smokers have developed lung cancer, compared to 90.25% of the non smokers.

  Moreover, within each of two subgroups (**tar**/no tar), **smokers** show a much lower percentage (**15%**, 10%) of cancer than **non smokers** (**95%**, 90%). (These numbers are contrary to empirical observations but well illustrate our point that observations are not to be trusted.)



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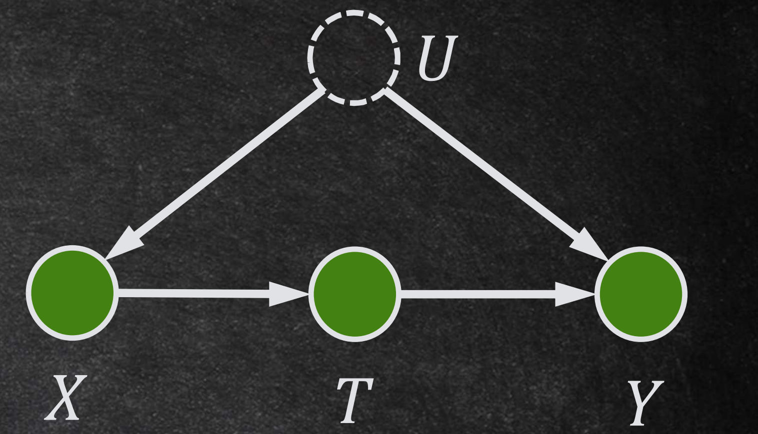

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Two opposing interpretations can be offered for these data.

- However, the antismoking lobbyists argue that the table tells an entirely different story—that smoking would actually increase, not decrease, one's risk of lung cancer.

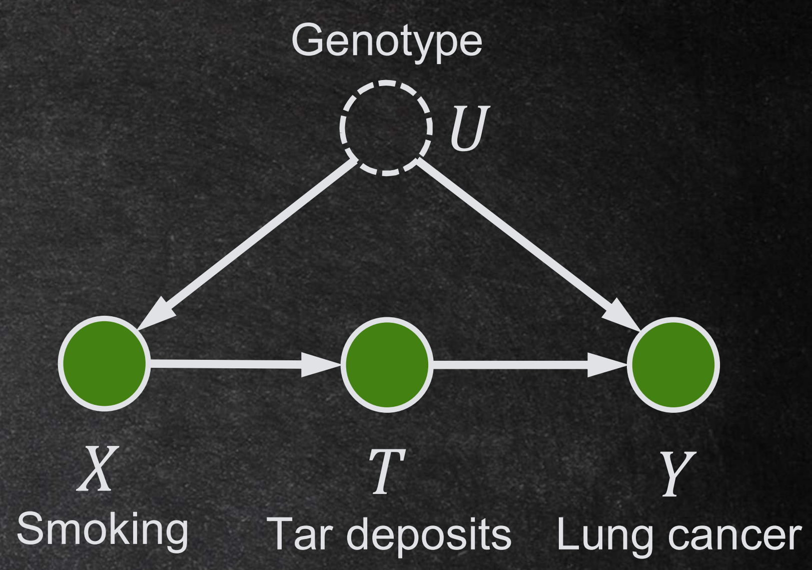

Figure 7.2

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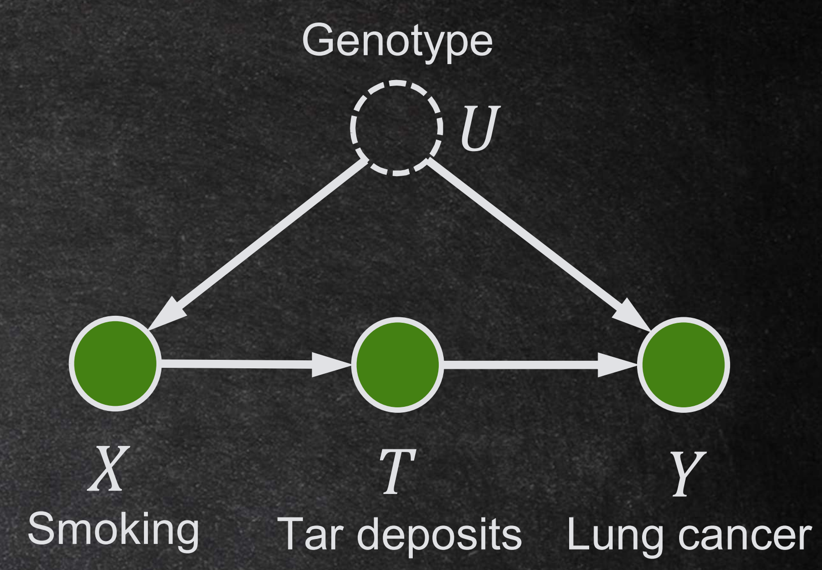
- However, the antismoking lobbyists argue that the table tells an entirely different story—that smoking would actually increase, not decrease, one's risk of lung cancer.
  - If you choose to smoke (**smokers**), then your chances of building up tar deposits are **95%** (380/400), compared to **5%** (20/400) if you choose not to smoke (**non smokers**).



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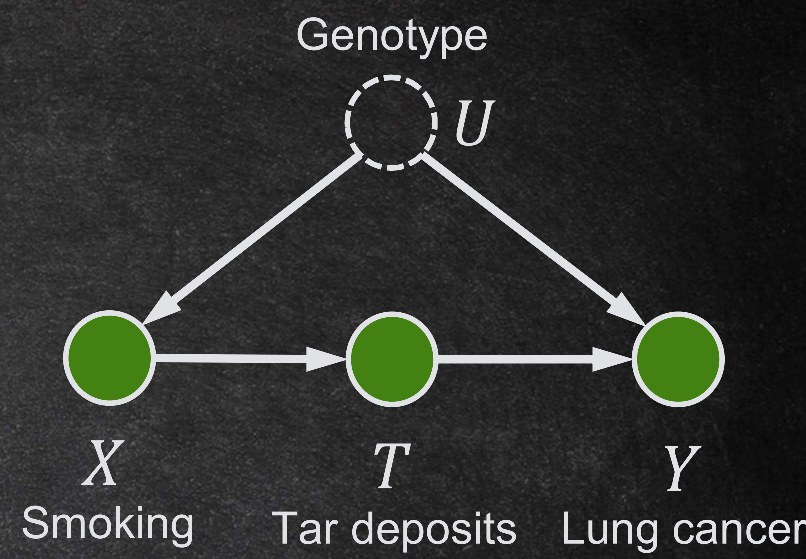
  - If you choose to smoke (smokers), then your chances of building up tar deposits are 95% (380/400), compared to 5% (20/400) if you choose not to smoke (non smokers).
  - To evaluate the effect of tar deposits, we look separately at two groups, **smokers** and **non smokers**, as done in the table below.



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	smokers		**non smokers**		**all subjects**	
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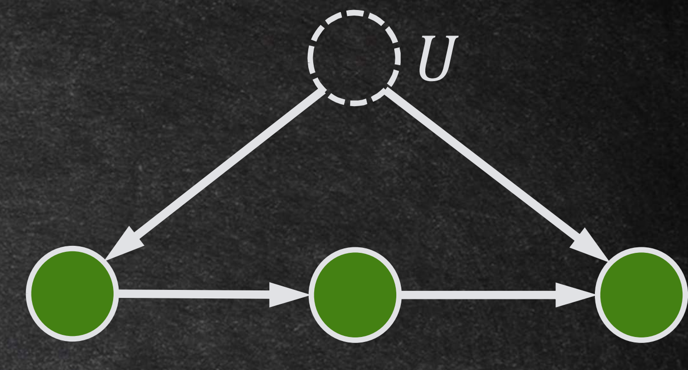
It appears that Tar deposits (T) have a harmful effect in both groups;

- in **smokers** it increases cancer rates from **10%** to **15%**, and



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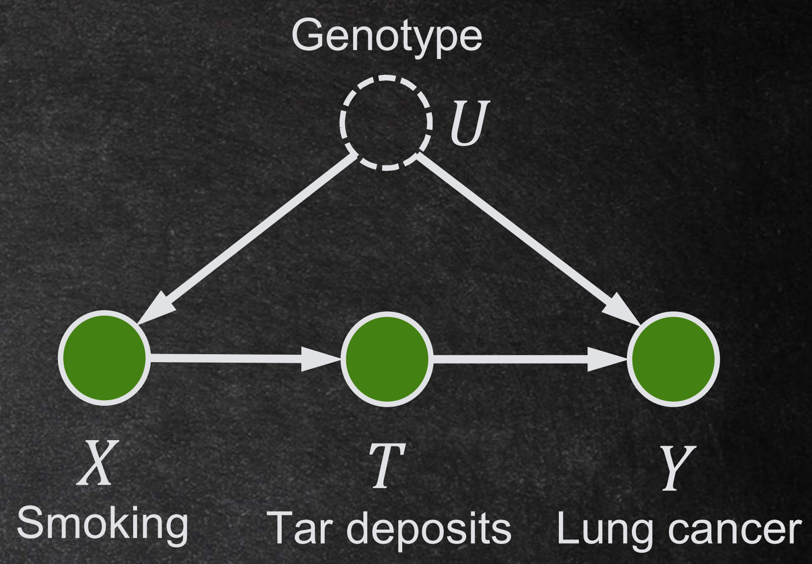

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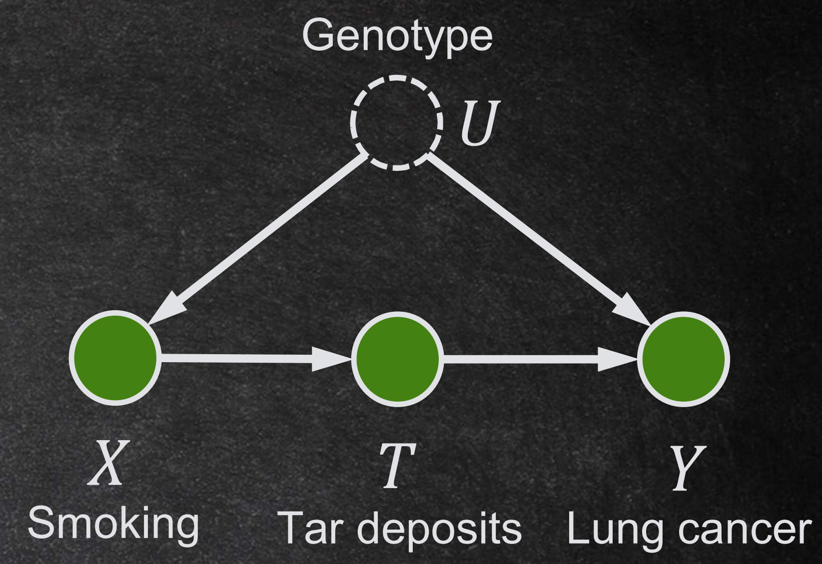
- in smokers it increases cancer rates from 10% to 15%, and
- in non smokers it increases cancer rates from 90% to 95%.

Thus, regardless of whether I have a natural craving for nicotine, I should avoid the harmful effect of tar deposits, and no-smoking offers very effective means of avoiding them.



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The graph of Figure 7.2 enables us to decide between these two groups of statisticians.

- First, we note that when the **ADJUSTMENT SET** $\mathbf{S}$ is the empty set, i.e., $\mathbf{S} = \{\emptyset\}$, the effect of X on T is identifiable, since there is no unblocked backdoor path from X to T.

THE BACKDOOR ADJUSTMENT

Given the modularity assumption, that, $\mathbf{S}$ satisfies the backdoor criterion, and positivity, we can identify the causal effect of X on Y as follows:

$$P(Y = y|do(X = x)) = \sum_{\mathbf{s}} P(Y = y|X = x, \mathbf{S} = \mathbf{s})\, P(\mathbf{S} = \mathbf{s})$$

Thus, the **BACKDOOR ADJUSTMENT** allows us to write

$$P(T = t|do(X = x)) = P(T = t|X = x)$$

Notice that in this case T serves as the outcome in place of Y.

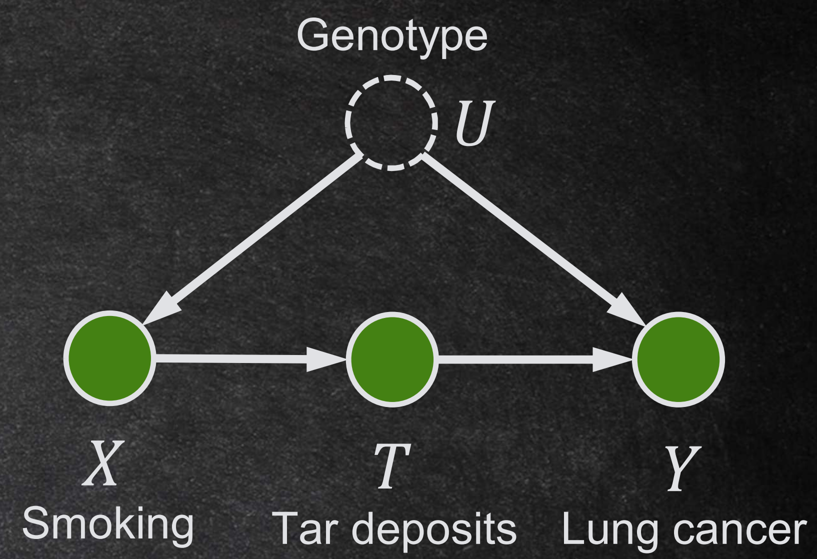

Figure 7.2

The backdoor path

$$X \leftarrow U \rightarrow Y \leftarrow T$$

is blocked at Y, which is a collider and does not belong to the adjustment set $\mathbf{S} = \{\emptyset\}$, i.e., $Y \notin \mathbf{S}$.

The graph of Figure 7.2 enables us to decide between these two groups of statisticians.

- Second, we note that the effect of T on Y is also identifiable, since the backdoor path from T to Y, namely

$$T \leftarrow X \leftarrow U \rightarrow Y$$

can be blocked by conditioning on X.

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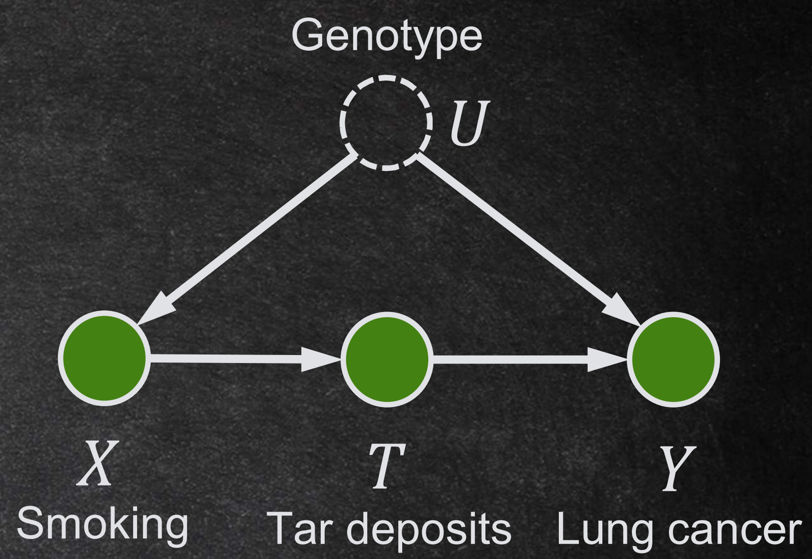

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If we let $\boldsymbol{S} = \{X\}$ be the **ADJUSTMENT SET**, then we can write the following

$$P\big(Y = y | do(T = t)\big) = \sum_{x'} P(Y = y | T = t, X = x')\, P(X = x')$$

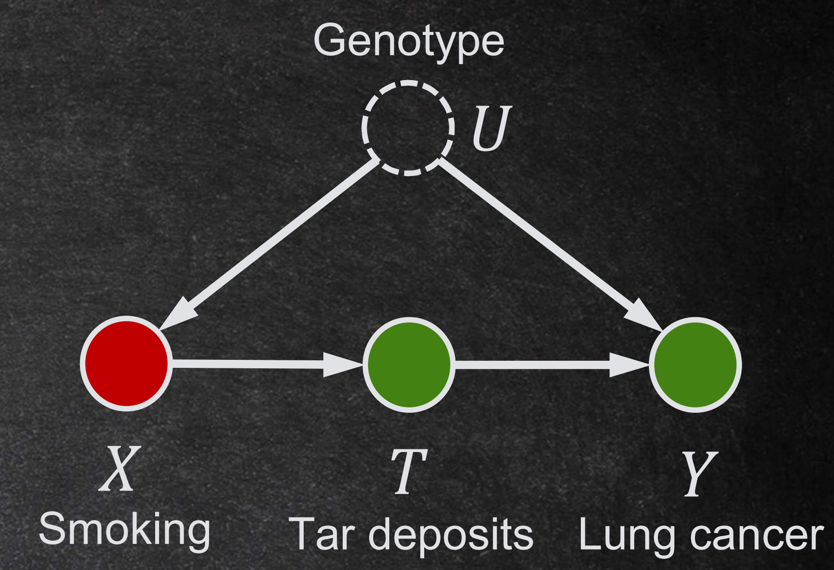


Figure 7.2

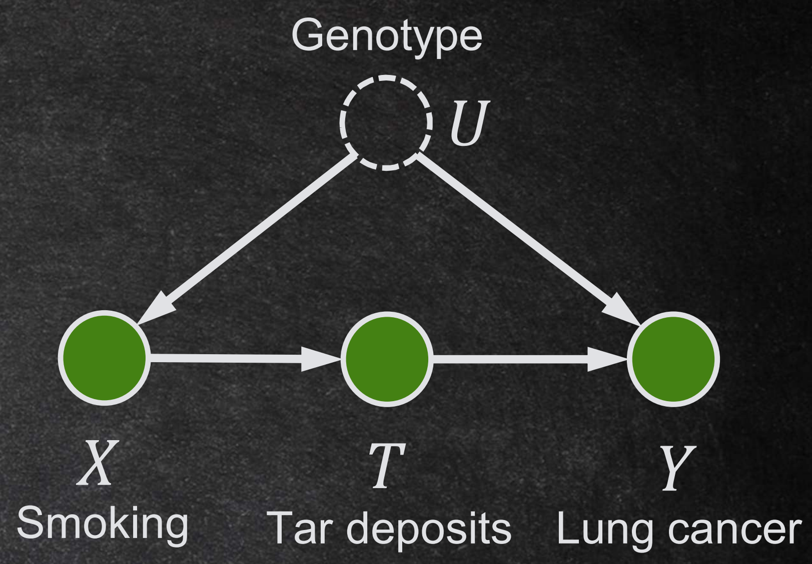
We are now going to chain together the two partial effects to obtain the overall effect of X on Y.

The reasoning goes as follows:

- If nature chooses to assign T the value t, then the probability of Y would be
  $$P\big(Y=y|do(T=t)\big)=\sum_{x'} P(Y=y|T=t,X=x')\,P(X=x')$$
- But the probability that nature would choose to do that (to set $T=t$), given that we choose to set X at x, is
  $$P\big(T=t|do(X=x)\big)=P(T=t|X=x)$$

Therefore, summing over all states t of T, we have

$$P\big(Y=y|do(X=x)\big)=\sum_{t} P\big(Y=y|do(T=t)\big)\,P\big(T=t|do(X=x)\big)$$
$$=\sum_{t}\sum_{x'} P(Y=y|T=t,X=x')\,P(X=x')\,P(T=t|X=x)$$



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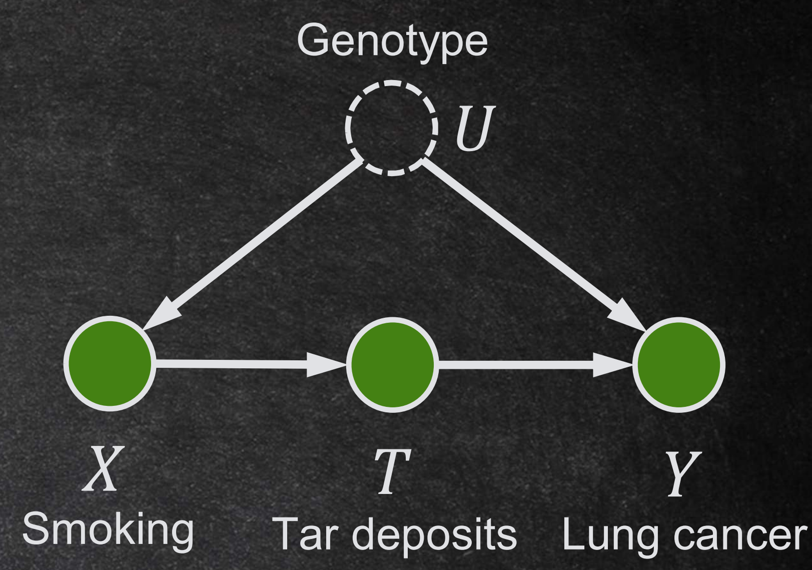

Figure 7.2

Applying the formula to the data in table above, we see that the tobacco industry was wrong;

- Tar deposits have a harmful effect in that they make lung cancer more likely and smoking, by increasing tar deposits, increases the chances of causing this harm.

$$P(Y = y|do(X = x)) = \sum_t P(Y = y|do(T = t))\, P(T = t|do(X = x))$$

$$= \sum_t \sum_{x\prime} P(Y = y|T = t, X = x')\, P(X = x')\, P(T = t|X = x)$$

The data in table above are obviously unrealistic and were deliberately crafted so as to surprise readers with counterintuitive conclusions that may emerge from naive analysis of observational data. In reality, we would expect observational studies to show positive correlation between smoking and lung cancer.

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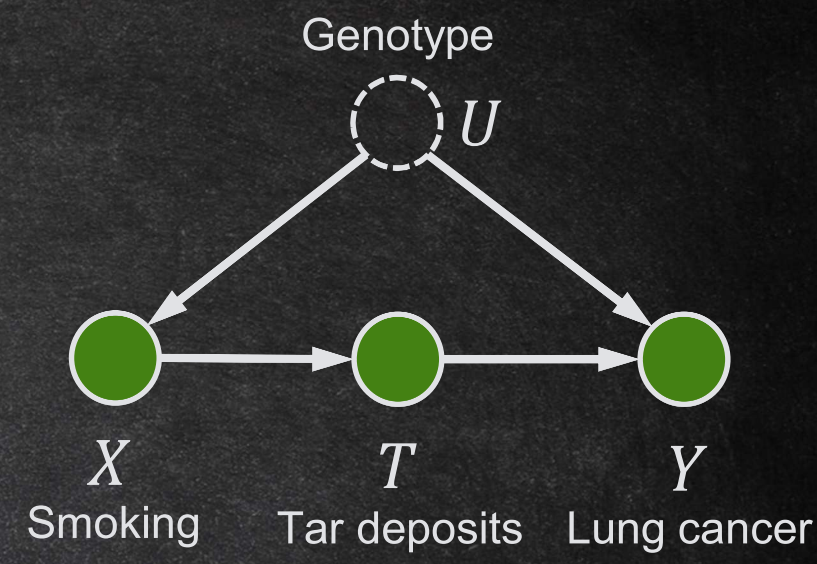

Figure 7.2

The **ESTIMAND**

$$P(Y = y | do(X = x)) = \sum_{t} P(T = t | X = x) \sum_{x'} P(Y = y | T = t, X = x')\, P(X = x')$$

could then be used for confirming and quantifying the harmful effect of smoking on cancer.

FRONTDOOR CRITERION

A set of variables $\mathbf{S}$ is said to satisfy the frontdoor criterion relative to an ordered pair of variables (X, Y) if

1. $\mathbf{S}$ intercepts all directed paths from X to Y.
2. There is no unblocked backdoor path from X to $\mathbf{S}$.
3. All backdoor paths from $\mathbf{S}$ to Y are blocked by X.

FRONTDOOR ADJUSTMENT

If $\mathbf{S}$ satisfies the frontdoor criterion relative to (X, Y) and if $P(x, \mathbf{s}) > 0$, then the causal effect of X on Y is identifiable and is given by the formula

$$P\big(y|do(x)\big) = \sum_{\mathbf{s}} P(\mathbf{s}|x) \sum_{x'} P(y|\mathbf{s}, x')\, P(x')$$

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Given an ordered pair of variables (X, Y) in a DAG $\mathcal{G}$, a set of variables $\mathbf{S}$ satisfies the backdoor criterion relative to (X, Y) if no node in $\mathbf{S}$ is a descendant of X, and $\mathbf{S}$ blocks every path between X and Y that contains an arrow into X.

Satisfying the **BACKDOOR CRITERION** isn't necessary to identify causal effects.

If the **FRONTDOOR CRITERION** is satisfied, that also gives us **IDENTIFIABILITY**.

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The conditions stated in the above definition are overly conservative; some of the backdoor paths excluded by conditions (2) and (3) can actually be allowed provided they are blocked by some variables.

There is a powerful symbolic machinery, called the **DO-CALCULUS**, that allows analysis of such intricate structures.

In fact, the **DO-CALCULUS** uncovers all causal effects that can be identified from a given graph.

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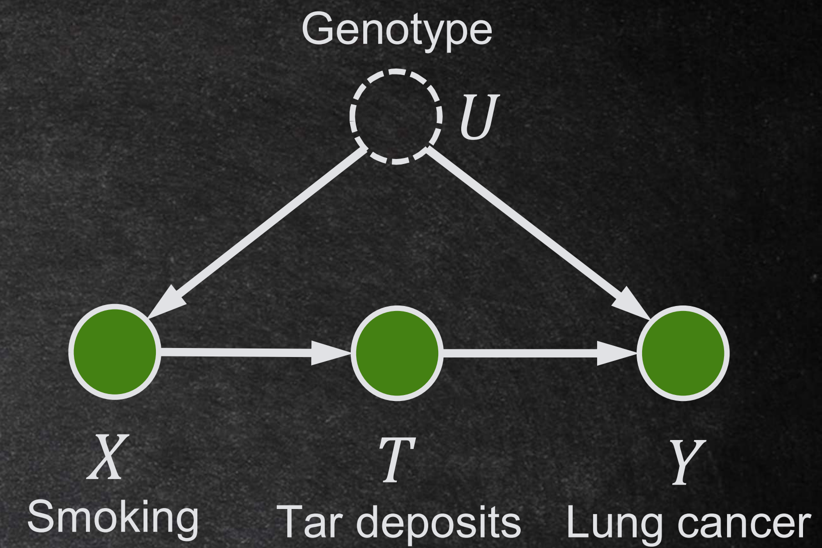
1. $\mathbf{S}$ intercepts all directed paths from X to Y.
2. There is no unblocked backdoor path from X to $\mathbf{S}$.
3. All backdoor paths from $\mathbf{S}$ to Y are blocked by X.

The combination of the adjustment formula, the backdoor criterion, and the frontdoor criterion covers numerous scenarios.

It proves the enormous, even revelatory, power that causal graphs have in not merely representing, but actually discovering causal information.

We now give a formal proof of the following

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_{x'} P(y|t,x')\, P(x')$$



We start from the joint probability, according to the Bayesian network factorization:

$$P(u,x,t,y) = P(x|u)\, P(t|x)\, P(y|u,t)\, P(u)$$

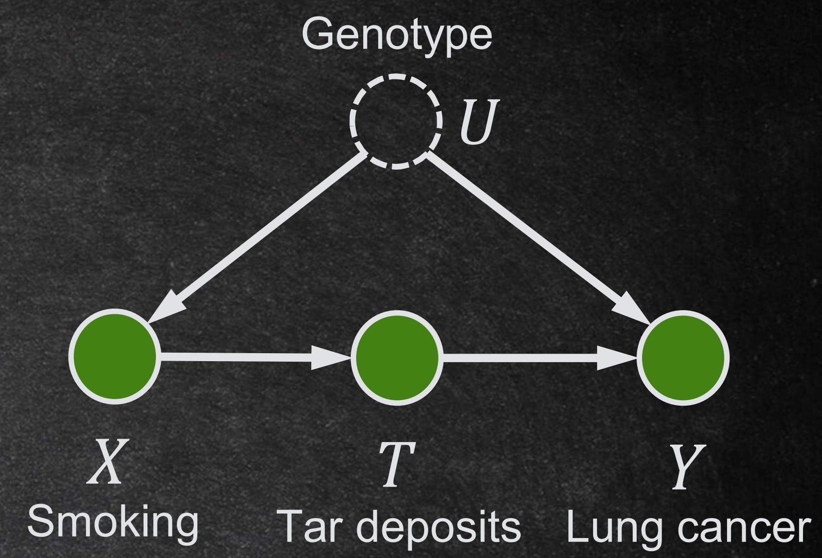
$$P\big(u,t,y|do(x)\big) = P(t|x)\, P(y|u,t)\, P(u)$$ (by truncated factorization)

$$\sum_t \sum_u P\big(u,t,y|do(x)\big) = \sum_t \sum_u P(t|x)\, P(y|u,t)\, P(u)$$ (marginalize out on t and u)

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_u P(y|u,t)\, P(u)$$

Even though we've removed all the *do* operators, recall that we are not done because, U is unobserved.

So we must also remove the u from the expression. This is where we have to get a bit creative.



$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_u P(y|u,t)\, P(u)$$

We want to be able to combine $P(y|u,t)$ and $P(u)$ into a joint factor over both y and u so that we can marginalize out u.

To do this, we need to get t behind the conditioning bar of the $P(u)$ factor.

This would be easy if we could just swap $P(u)$ out for $P(u|t)$ in the above equation.

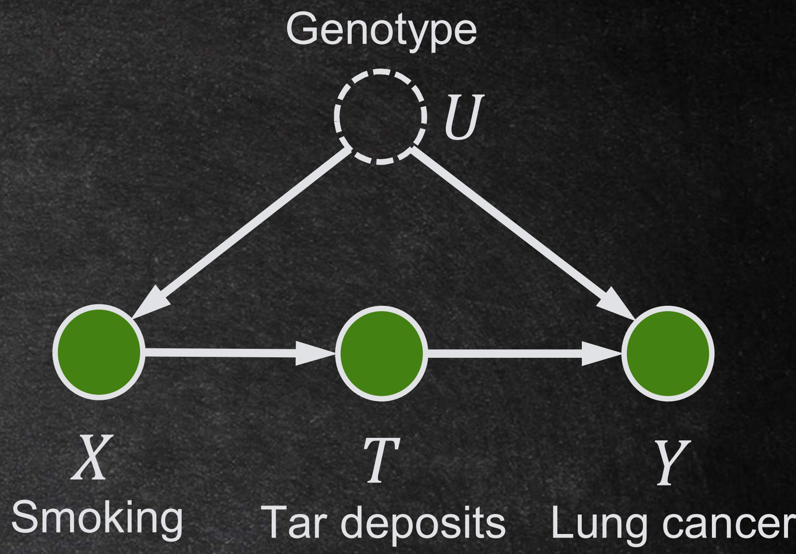
The key thing to notice is that we actually can include t behind the conditioning bar if x were also there because X *d*-separates, U from T.

In math, this means that the following equality holds:

$$P(u|x) = P(u|x,t)$$

To achieve what we want, we exploit the usual trick of conditioning on it (x) and marginalizing it out.

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_u P(y|u,t)\, P(u)$$



Thus, conditioning on x and marginalizing it out gives

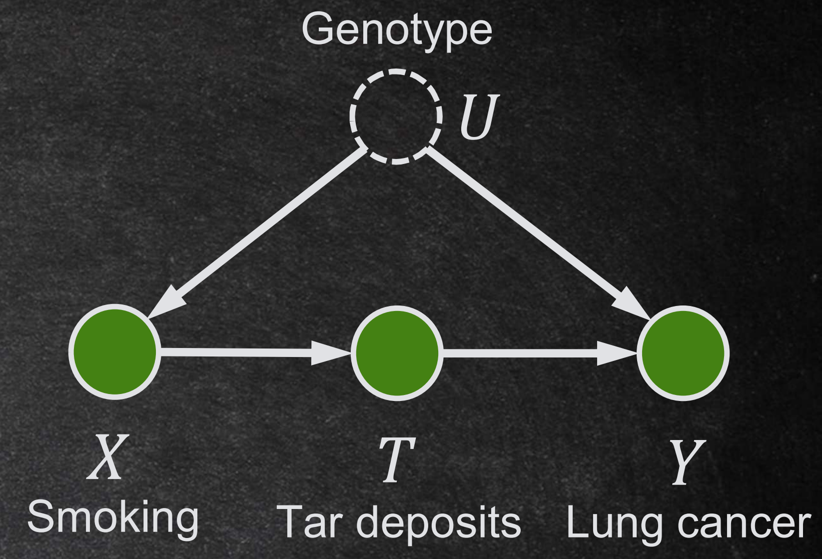
$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_u P(y|u,t) \sum_{x'} P(u|x')\, P(x')$$

$$= \sum_t P(t|x) \sum_u P(y|u,t) \sum_{x'} P(u|x',t)\, P(x')$$

$$= \sum_t P(t|x) \sum_{x'} P(x') \sum_u P(y|u,t)\, P(u|x',t)$$

Great, but now we can't combine $P(y|u,t)$ and $P(u|x',t)$ because $P(y|u,t)$ is missing this newly introduced x' behind its conditioning bar.

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_{x'} P(x') \sum_u P(y|u,t)\, P(u|x',t)$$



We can solve the issue to obtain the following:

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_{x'} P(x') \sum_u P(y|u,x',t)\, P(u|x',t)$$

$$= \sum_t P(t|x) \sum_{x'} P(x') \sum_u P(y,u|x',t)$$

$$= \sum_t P(t|x) \sum_{x'} P(y|x',t)\, P(x')$$

We've completed the derivation of the frontdoor adjustment without using the backdoor adjustment.

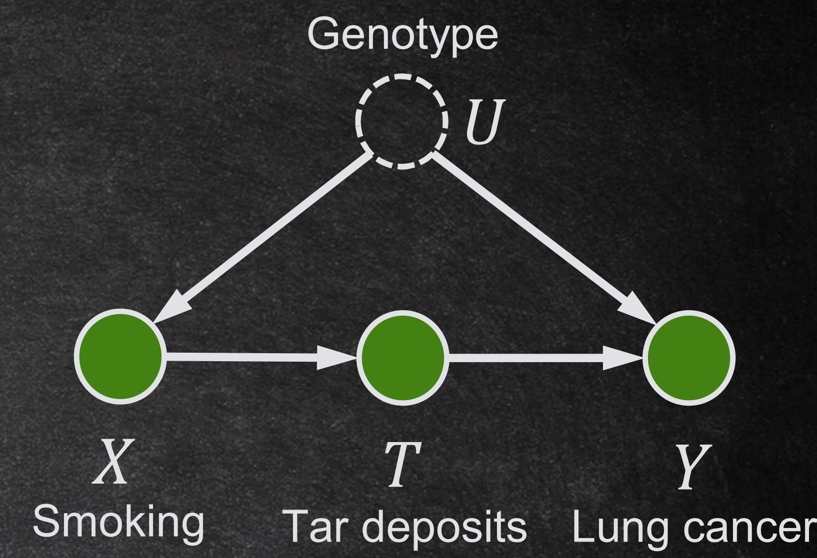
FRONTDOOR ADJUSTMENT

If $\boldsymbol{S}$ satisfies the frontdoor criterion relative to (X,Y) and if $P(x,\boldsymbol{s}) > 0$, then the causal effect of X on Y is identifiable and is given by the formula

$$P\big(y|do(x)\big) = \sum_{\boldsymbol{s}} P(\boldsymbol{s}|x) \sum_{x'} P(y|\boldsymbol{s},x')\, P(x')$$

However, we still need to show that

$$P\big(y|do(x)\big) = \sum_t P\big(t|do(x)\big)\, P\big(y|do(t)\big)$$



To do that, all that's left is to recognize that these parts match the following equations

$$P\big(t|do(x)\big) = P(t|x)$$

$$P\big(y|do(t)\big) = \sum_{x'} P(y|t,x')\, P(x')$$

which brings us to

$$P\big(y|do(x)\big) = \sum_t P(t|x) \sum_{x'} P(y|t,x')\, P(x')$$

FRONTDOOR ADJUSTMENT

If $\boldsymbol{S}$ satisfies the frontdoor criterion relative to (X, Y) and if $P(x, \boldsymbol{s}) > 0$, then the causal effect of X on Y is identifiable and is given by the formula

$$P\big(y|do(x)\big) = \sum_{\boldsymbol{s}} P(\boldsymbol{s}|x) \sum_{x'} P(y|\boldsymbol{s},x')\, P(x')$$

Part II
do-Calculus

We now ask the following questions:

- Can we identify causal estimands when the associated causal graph satisfies neither the backdoor criterion nor the frontdoor criterion?
- If so, how?

Pearl's **DO-CALCULUS** answers to these questions, and

- gives us tools to identify causal effects using the causal assumptions encoded in the causal graph,
- allows us to identify any causal estimand that is identifiable.

We can use **DO-CALCULUS** to identify causal effects where there are multiple treatments and/or multiple outcomes.

Consider an arbitrary **CAUSAL ESTIMAND**

$$P(\mathbf{Y}|do(\mathbf{X}=\boldsymbol{x}),\mathbf{Z}=\boldsymbol{z})$$

$\mathbf{Y}$ is a set of outcome variables

$\mathbf{X}$ is a set of treatment variables

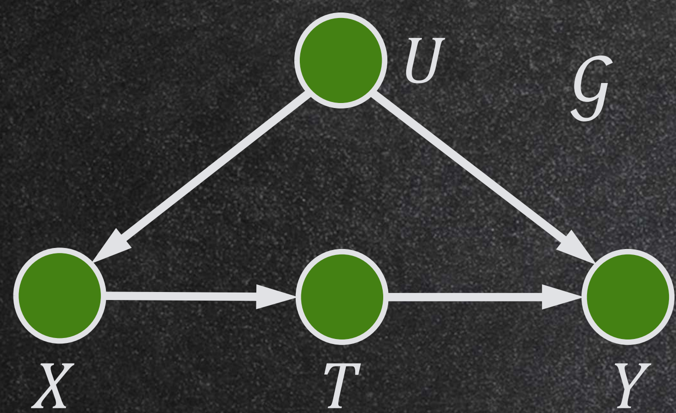
$\mathbf{Z}$ is a set of covariates (empty set included)

To introduce and discuss do-calculus we first need to give some more notation.

It is useful to remember that we use

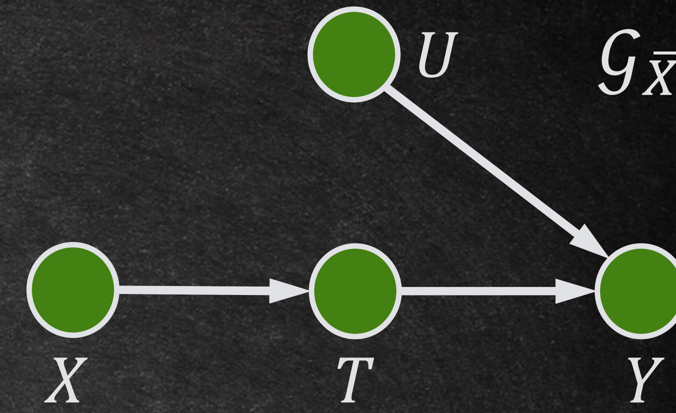
$$\perp\!\!\!\perp_{\mathcal{G}}$$

to denote *d*-separation in $\mathcal{G}$.

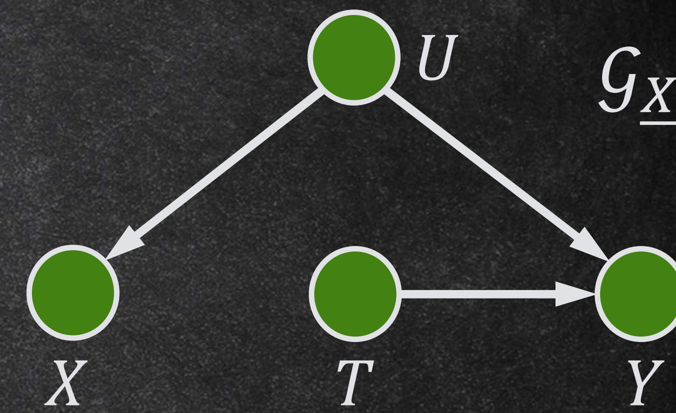


We're now ready; do-calculus consists of just three rules.

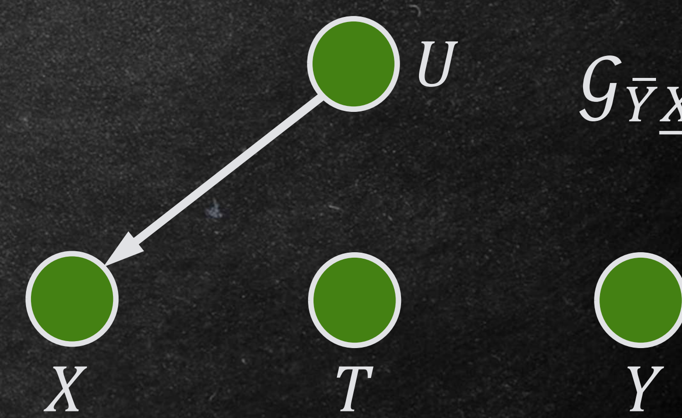
The graph $\mathcal{G}$ where all incoming edges in X have been removed.



The graph $\mathcal{G}$ where all outgoing edges from X have been removed.



The graph $\mathcal{G}$ where all incoming edges in Y and all outgoing edges from X have been removed.



RULES OF DO-CALCULUS

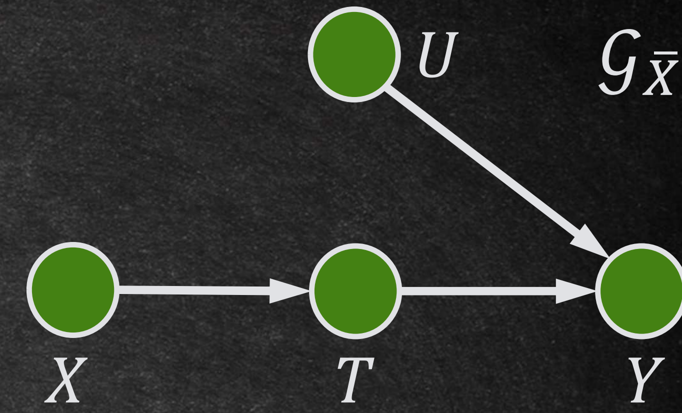
Given a causal graph $\mathcal{G}$, an associated distribution P, and disjoint sets of variables $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$, the following rules hold:

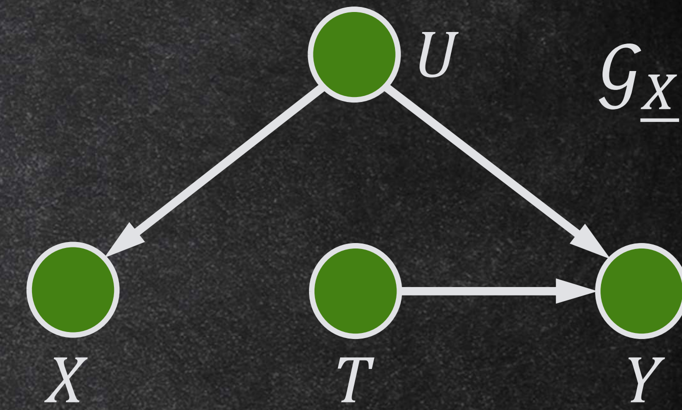
Rule 1: $P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

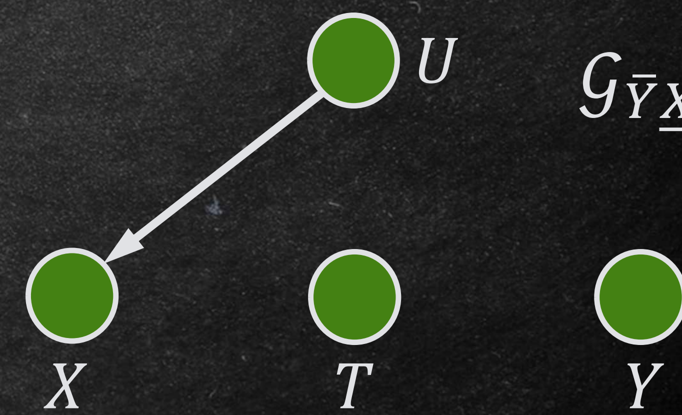
Rule 2: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 3: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\,\overline{\mathbf{Z}(\mathbf{W})}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

where $\overline{\mathbf{Z}(\mathbf{W})}$ denotes the set of nodes of $\mathbf{Z}$ that aren't ancestors of any node of $\mathbf{W}$ in $\mathcal{G}_{\overline{X}}$.







In the next slides, rather than recreating the proofs for these rules from Pearl, we'll give intuition for each of them in terms of concepts we've already presented in this course.

RULE 1: $P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$ $\quad \mathbf{S} = \{\mathbf{W}\}$

Simply removing the intervention $do(\boldsymbol{x})$, we get the following

$$P(\boldsymbol{y}|\boldsymbol{z}, \boldsymbol{w}) = P(\boldsymbol{y}|\boldsymbol{w}) \quad \text{if } \mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}} \mathbf{Z} \mid \mathbf{W}$$

(This is what d-separation gives us under the Markov assumption)

From the **GLOBAL MARKOV ASSUMPTION**, d-separation in the graph $\mathcal{G}$ implies conditional independence in P.

This means that **RULE 1** is simply a generalization of d-separation to interventional distributions.

GLOBAL MARKOV ASSUMPTION

Given that P is Markov with respect to $\mathcal{G}$ (satisfies the local Markov assumption), if X and Y are d-separated in $\mathcal{G}$ conditioned on $\mathbf{S}$, then X and Y are independent in P conditioned on $\mathbf{S}$. We can write this succinctly as follows:

$$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid \mathbf{S} \Rightarrow X \perp\!\!\!\perp_{P} Y \mid \mathbf{S}$$

D-SEPARATION

A path p is blocked by a set of nodes $\mathbf{S} \subseteq \{1,2, \dots, n\}$ if and only if

1) p contains a chain of nodes $A \longrightarrow B \longrightarrow C$ or a fork $A \longleftarrow B \longrightarrow C$ such that the middle node B is in $\mathbf{S}$ (i.e., is conditioned on),
2) or p contains a collider $A \longrightarrow B \longleftarrow C$ such that the collision node B is not in $\mathbf{S}$, and no descendant of B is in $\mathbf{S}$.

If $\mathbf{S}$ blocks every path between two nodes X and Y, then X and Y are d-separated, conditional on $\mathbf{S}$, and thus are independent conditional on $\mathbf{S}$.

RULE 2: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$ $\quad \mathbf{S} = \{\mathbf{W}\}$

Simply removing the intervention $do(\boldsymbol{x})$, we get the following

$$P(\boldsymbol{y}|do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|\boldsymbol{z}, \boldsymbol{w}) \quad \text{if } \mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{W}$$

(This is what we do when we justify the backdoor adjustment using the backdoor criterion)

Association is causation if the outcome Y and the treatment Z are d-separated by some set of variables $\mathbf{W}$ that are conditioned on.

RULE 2 is a generalization of the backdoor adjustment to interventional distributions.

THE BACKDOOR CRITERION

Given an ordered pair of variables (X, Y) in a DAG $\mathcal{G}$, a set of variables $\mathbf{S}$ satisfies the backdoor criterion relative to (X, Y) if no node in $\mathbf{S}$ is a descendant of X, and $\mathbf{S}$ blocks every path between X and Y that contains an arrow into X.

THE BACKDOOR ADJUSTMENT

Given the modularity assumption, that, $\mathbf{S}$ satisfies the backdoor criterion, and positivity, we can identify the causal effect of X on Y as follows:

$$P\big(Y = y|do(X = x)\big) = \sum_{\boldsymbol{s}} P(Y = y|X = x, \mathbf{S} = \boldsymbol{s})\, P(\mathbf{S} = \boldsymbol{s})$$

RULE 3: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\overline{\mathbf{Z}(\mathbf{W})}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

We do not discuss further **RULE 3**, because its' proof is rather technical.

We could ask whether there could exist causal estimands that are identifiable but that can't be identified using only the rules of do-calculus in theorem below.

RULES OF DO-CALCULUS

Given a causal graph $\mathcal{G}$, an associated distribution P, and disjoint sets of variables $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$, the following rules hold:

Rule 1: $P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 2: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 3: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\overline{\mathbf{Z}(\mathbf{W})}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

where $\overline{\mathbf{Z}(\mathbf{W})}$ denotes the set of nodes of $\mathbf{Z}$ that aren't ancestors of any node of $\mathbf{W}$ in $\mathcal{G}_{\overline{X}}$.

Fortunately, it has been proved that **DO-CALCULUS IS COMPLETE**, which means that these three rules are sufficient to identify all identifiable causal estimands.

Because these proofs are constructive, they also admit algorithms that identify any causal estimand in polynomial time.

Note that all of this is about **NONPARAMETRIC IDENTIFICATION**; in other words, do-calculus tells us if we can identify a given causal estimand using only the causal assumptions encoded in the causal graph.

If we introduce more assumptions about the distribution (e.g. linearity), we can identify more causal estimands.

That would be known as **PARAMETRIC IDENTIFICATION**.

RULES OF DO-CALCULUS

Given a causal graph $\mathcal{G}$, an associated distribution P, and disjoint sets of variables $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$, the following rules hold:

Rule 1: $P(\boldsymbol{y}|do(\boldsymbol{x}),\boldsymbol{z},\boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}),\boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 2: $P(\boldsymbol{y}|do(\boldsymbol{x}),do(\boldsymbol{z}),\boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}),\boldsymbol{z},\boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 3: $P(\boldsymbol{y}|do(\boldsymbol{x}),do(\boldsymbol{z}),\boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}),\boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\overline{\mathbf{Z}(\mathbf{W})}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

where $\overline{\mathbf{Z}(\mathbf{W})}$ denotes the set of nodes of $\mathbf{Z}$ that aren't ancestors of any node of $\mathbf{W}$ in $\mathcal{G}_{\overline{X}}$.

Let us now apply the rules of do-calculus to the graph in Figure 7.2.

The frontdoor adjustment gives

$$P\big(y|do(x)\big) = \sum_{t} P(t|x) \sum_{x'} P(y|t,x')\, P(x')$$

We'll now do the frontdoor adjustment proof using the rules of do-calculus.

Our goal is to identify $P\big(y|do(x)\big)$

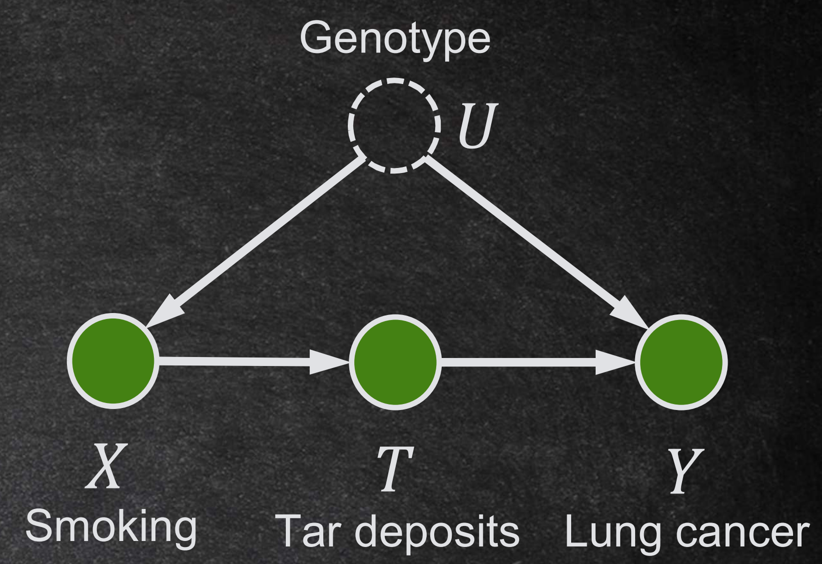


Figure 7.2

Part III

Determining Identifiability from the Graph

We previously mentioned that **DO-CALCULUS IS COMPLETE**, which means that three rules are sufficient to identify all identifiable causal estimands.

However, it would be much more satisfying to know whether a causal estimand is identifiable by simply looking at the causal graph.

RULES OF DO-CALCULUS

Given a causal graph $\mathcal{G}$, an associated distribution P, and disjoint sets of variables $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{W}$, the following rules hold:

Rule 1: $P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 2: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{z}, \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

Rule 3: $P(\boldsymbol{y}|do(\boldsymbol{x}), do(\boldsymbol{z}), \boldsymbol{w}) = P(\boldsymbol{y}|do(\boldsymbol{x}), \boldsymbol{w})$ if $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\overline{\mathbf{Z}(\mathbf{W})}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W}$

where $\overline{\mathbf{Z}(\mathbf{W})}$ denotes the set of nodes of $\mathbf{Z}$ that aren't ancestors of any node of $\mathbf{W}$ in $\mathcal{G}_{\overline{X}}$.

We previously mentioned that **DO-CALCULUS IS COMPLETE**, which means that three rules are sufficient to identify all identifiable causal estimands.

However, it would be much more satisfying to know whether a causal estimand is identifiable by simply looking at the causal graph.

For example, the **BACKDOOR CRITERION** and the **FRONTDOOR CRITERION** gave us simple ways to know for sure that a causal estimand is identifiable.

However, there are plenty causal estimands that are identifiable, even though the corresponding causal graphs don't satisfy the backdoor or frontdoor criterion.

More general graphical criteria exist that will tell us that these estimands are identifiable.

THE BACKDOOR CRITERION

Given an ordered pair of variables (X, Y) in a DAG $\mathcal{G}$, a set of variables $\boldsymbol{S}$ satisfies the backdoor criterion relative to (X, Y) if no node in $\boldsymbol{S}$ is a descendant of X, and $\boldsymbol{S}$ blocks every path between X and Y that contains an arrow into X.

FRONTDOOR CRITERION

A set of variables $\boldsymbol{S}$ is said to satisfy the frontdoor criterion relative to an ordered pair of variables (X, Y) if

1. $\boldsymbol{S}$ intercepts all directed paths from X to Y.
2. There is no unblocked backdoor path from X to $\boldsymbol{S}$.
3. All backdoor paths from $\boldsymbol{S}$ to Y are blocked by X.

When we care about causal effects of an intervention on a single variable, Tian and Pearl provide a relatively simple graphical criterion that is sufficient for identifiability: the **UNCONFOUNDED CHILDREN CRITERION**.

This criterion generalizes the backdoor criterion and the frontdoor criterion.

Like backdoor criterion and the frontdoor criterion, **UNCONFOUNDED CHILDREN CRITERION** is a sufficient condition for identifiability:

UNCONFOUNDED CHILDREN IDENTIFIABILITY

Let $\mathbf{Y}$ be the set of outcome variables and X be a single variable. If the unconfounded children criterion and positivity are satisfied, then

$$P(\mathbf{Y} = \mathbf{y} | do(x))$$

is identifiable.

UNCONFOUNDED CHILDREN CRITERION

This criterion is satisfied if it is possible to block all backdoor paths from the treatment variable X to all of its children ($ch(X)$) that are ancestors ($an(\mathbf{Y})$) of $\mathbf{Y}$ with a single conditioning set $\mathbf{S}$.

⇓

IDENTIFIABILITY

The intuition for this is similar to that of the frontdoor criterion, i.e., if we can isolate all of the causal association flowing out of treatment X along directed paths to $\mathbf{Y}$, we have identifiability.

1. First, consider that all of the causal association from X to $\mathbf{Y}$ must flow through its children $ch(X)$.

2. We can isolate this causal association if there is no confounding between X and any of its children $ch(X)$.

3. This isolation of all of the causal association is what gives us identifiability of the causal effect of X on any other node in the graph.

4. This intuition might lead you to suspect that this criterion is necessary in the very specific case where the outcome set $\mathbf{Y}$ is all of the other variables in the graph other than X; it turns out that this is true.

   But this condition is not necessary if $\mathbf{Y}$ is a smaller set than that.

UNCONFOUNDED CHILDREN CRITERION

This criterion is satisfied if it is possible to block all backdoor paths from the treatment variable X to all of its children ($ch(X)$) that are ancestors ($an(\mathbf{Y})$) of $\mathbf{Y}$ with a single conditioning set $\mathbf{S}$.

⇓

IDENTIFIABILITY

The intuition for this is similar to that of the frontdoor criterion, i.e., if we can isolate all of the causal association flowing out of treatment X along directed paths to $\mathbf{Y}$, we have identifiability.

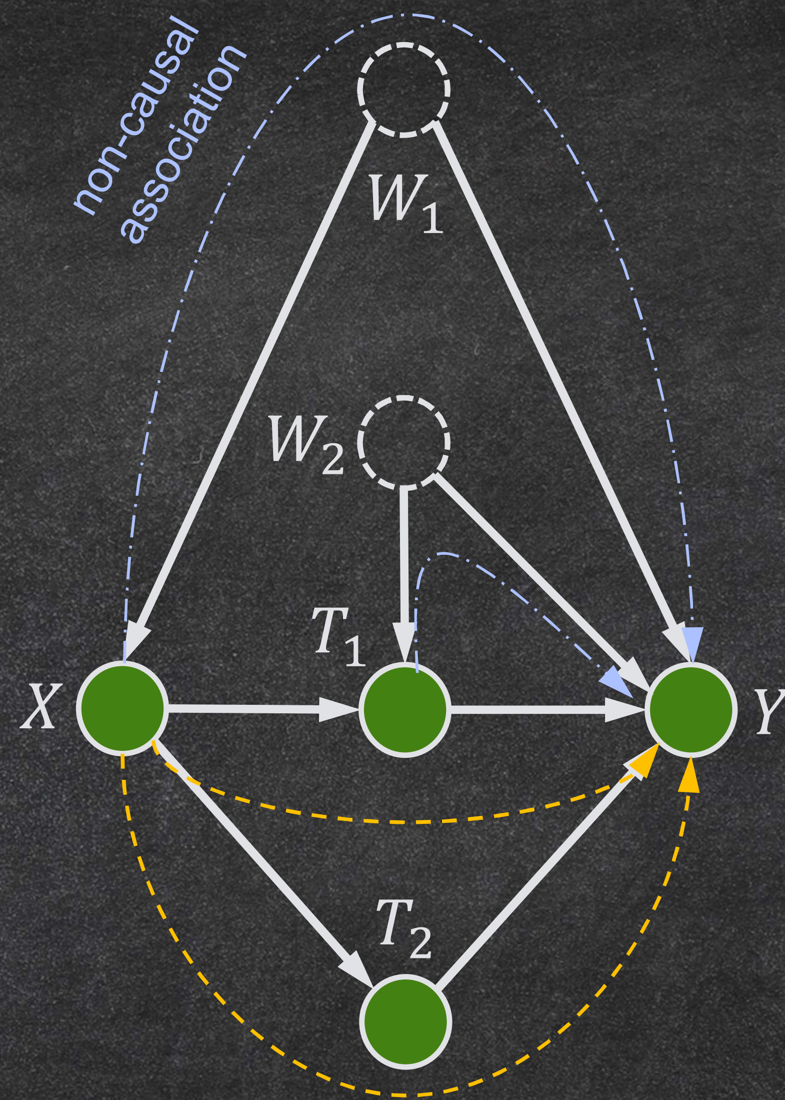


Figure 7.5(a)

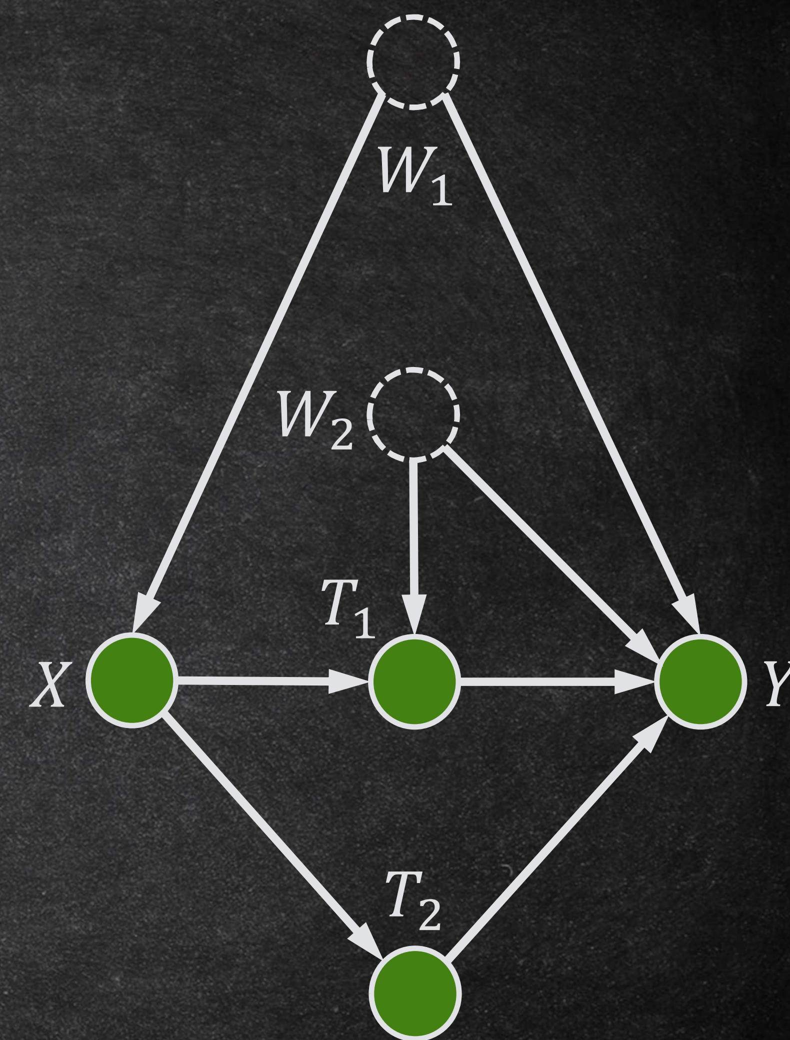


Figure 7.5(b)

1. First, consider that all of the causal association from X to $\mathbf{Y}$ must flow through its children $ch(X) = \{T_1, T_2\}$.

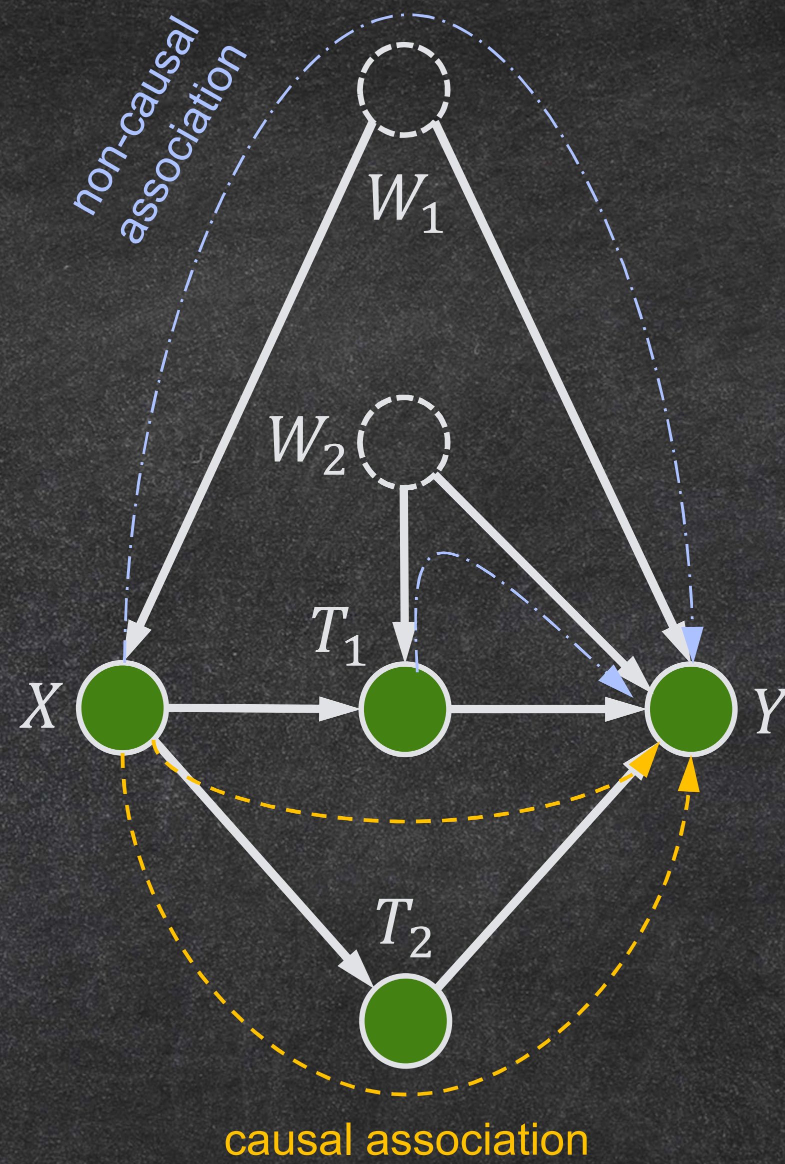


Figure 7.5(a)

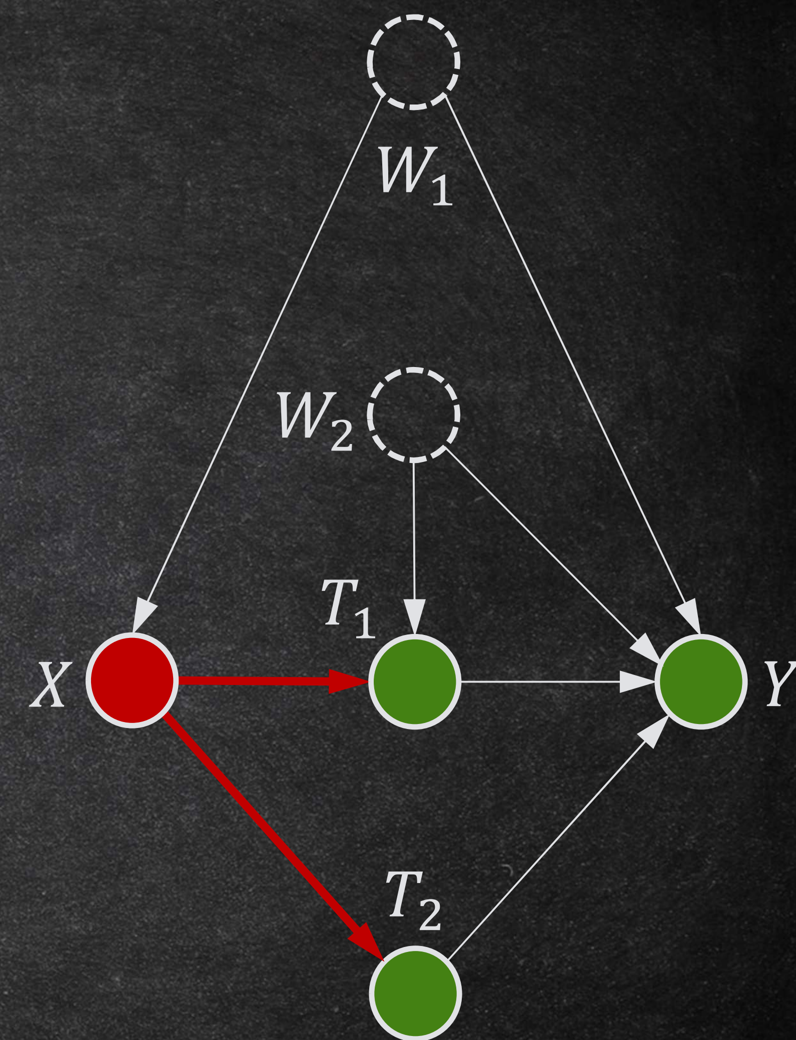


Figure 7.5(c)

2. We can isolate this causal association if there is no confounding between X and any of its children $ch(X) = \{T_1, T_2\}$.

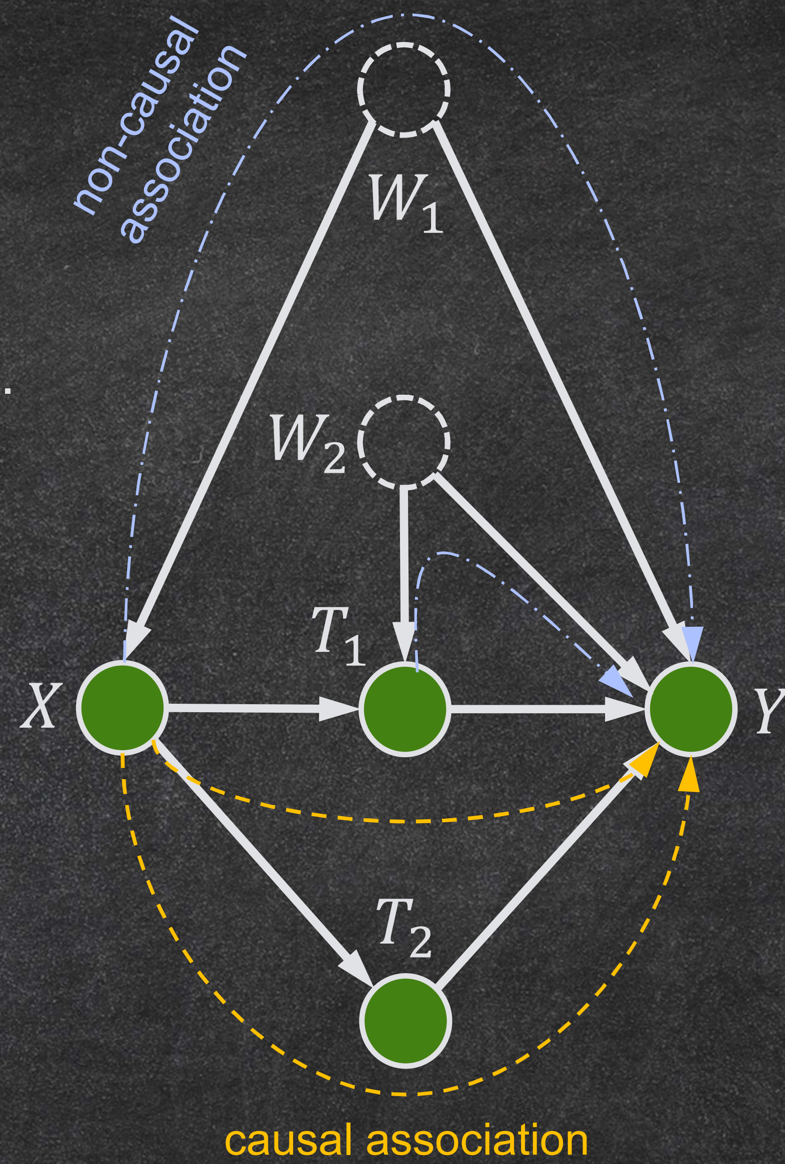


Figure 7.5(a)

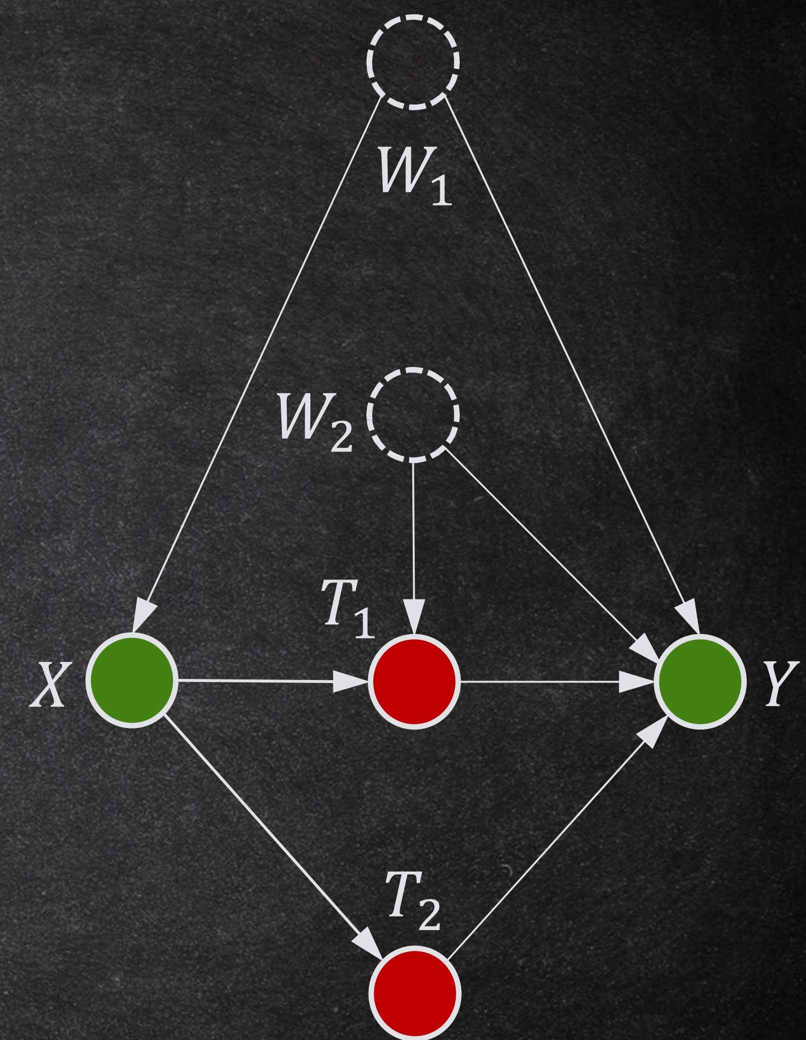


Figure 7.5(d)

2. We can isolate this causal association if there is no confounding between X and any of its children $ch(X) = \{T_1, T_2\}$.

$X \leftarrow W_1 \rightarrow Y \leftarrow T_1$

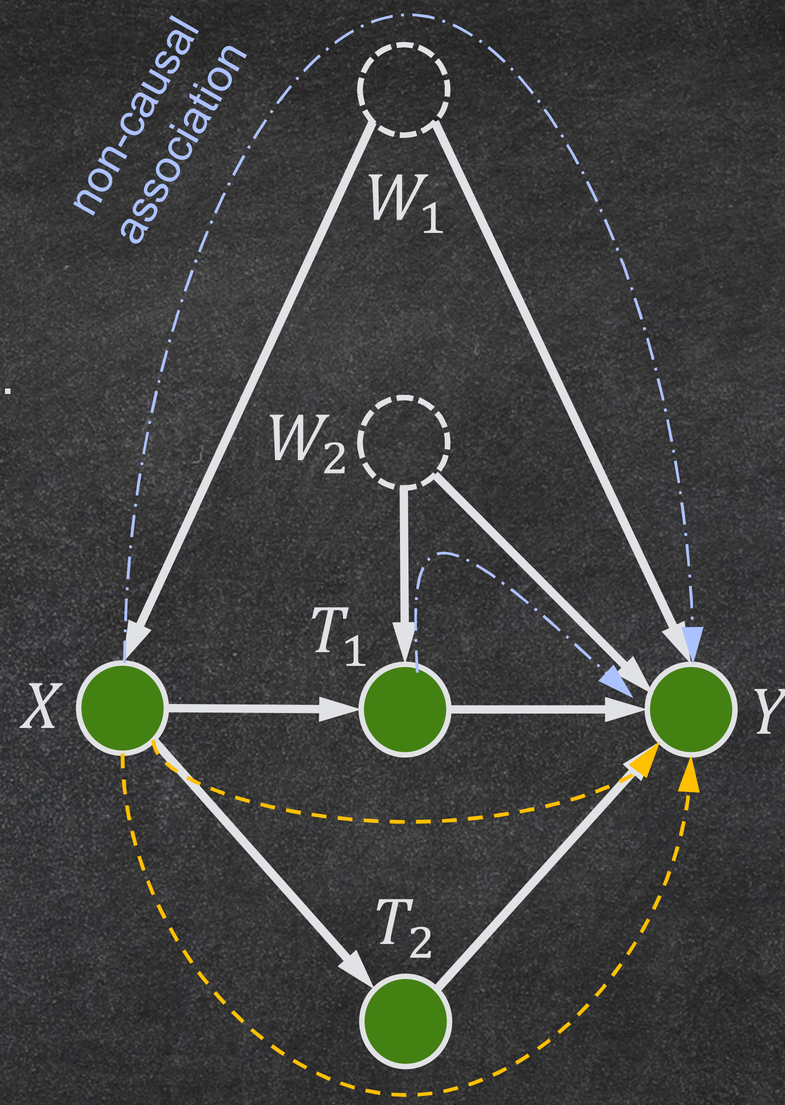


Figure 7.5(a)

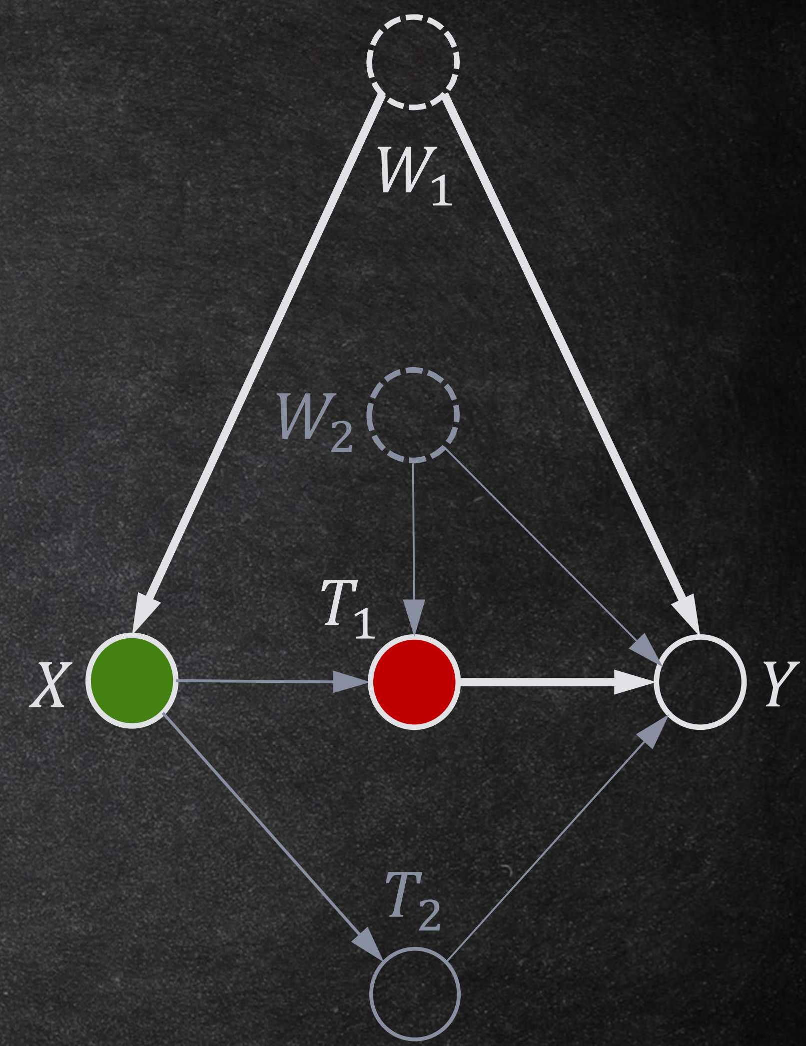


Figure 7.5(e)

2. We can isolate this causal association if there is no confounding between X and any of its children $ch(X) = \{T_1, T_2\}$.

$X \leftarrow W_1 \rightarrow Y \leftarrow T_1$

$X \leftarrow W_1 \rightarrow Y \leftarrow T_2$

blocked backdoor paths!!!

Y is a collider and does not belong to the adjustment set

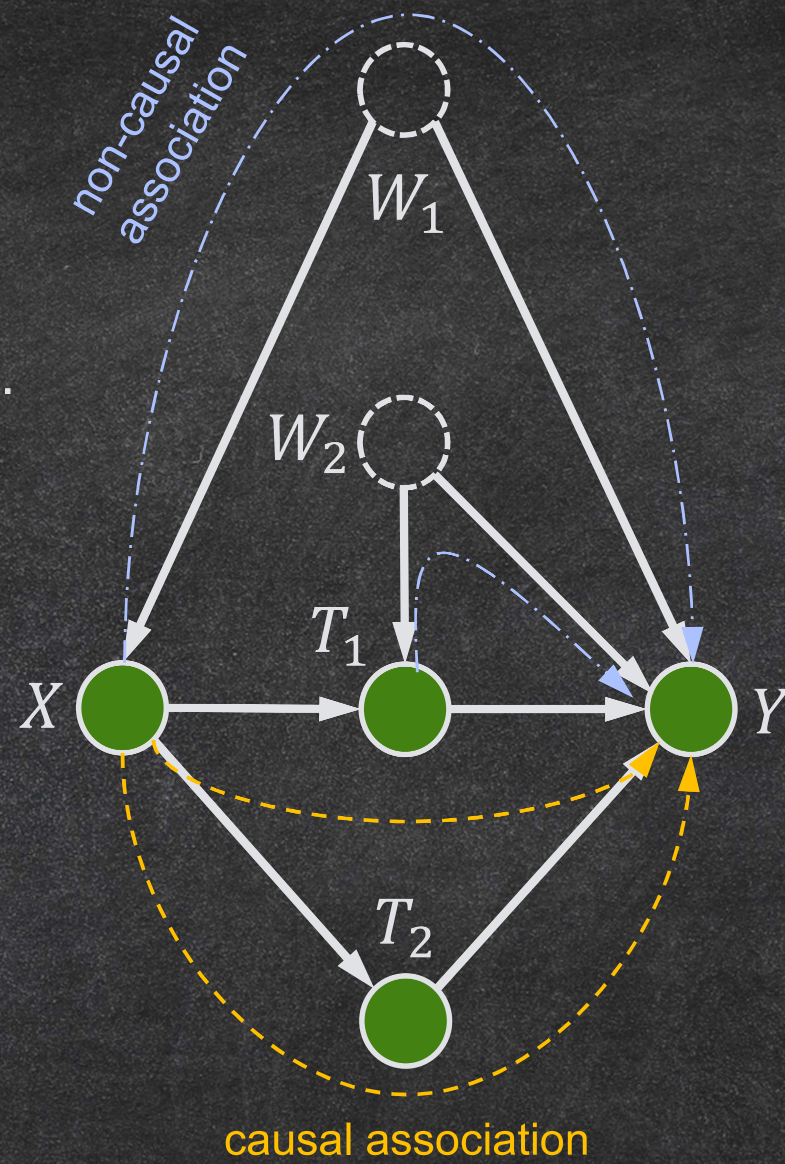


Figure 7.5(a)

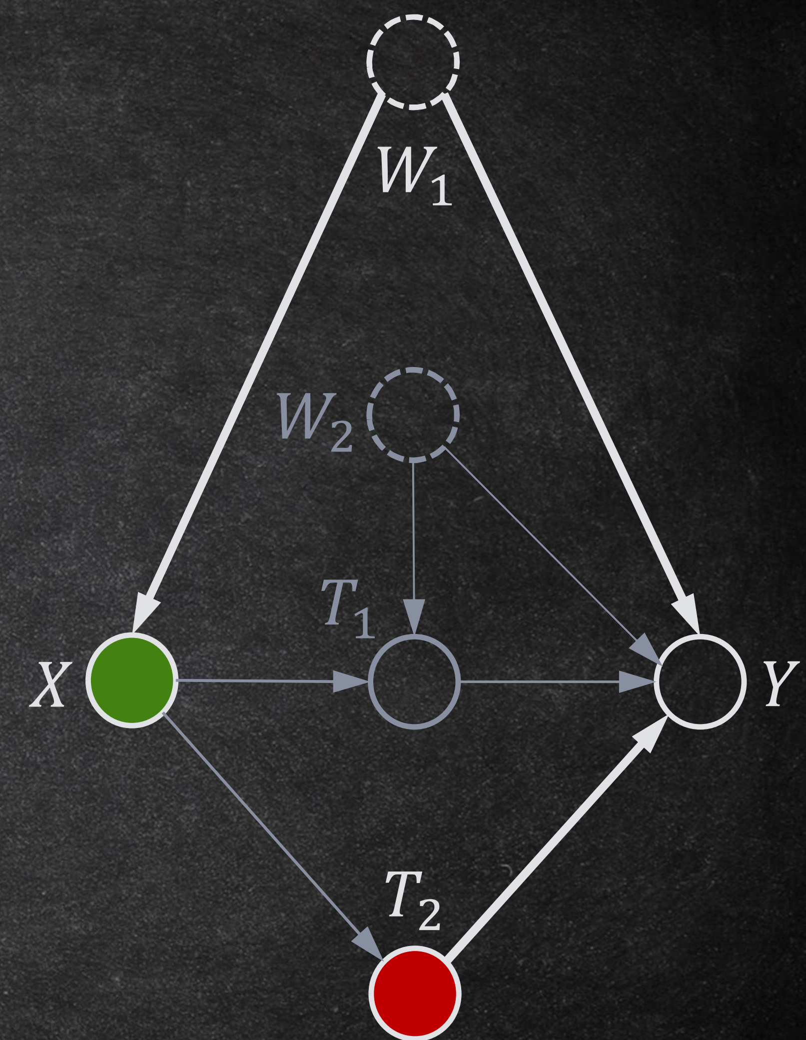


Figure 7.5(f)

We already clarified that the **UNCONFOUNDED CHILDREN CRITERION** is not necessary for identifiability, but it might aid your graphical intuition to have a **NECESSARY CONDITION** in mind.

A concrete example is as follows:

- for each backdoor path from X to any child $T \in ch(X)$ of X that is an ancestor $an(Y)$ of Y, it is possible to block that path.

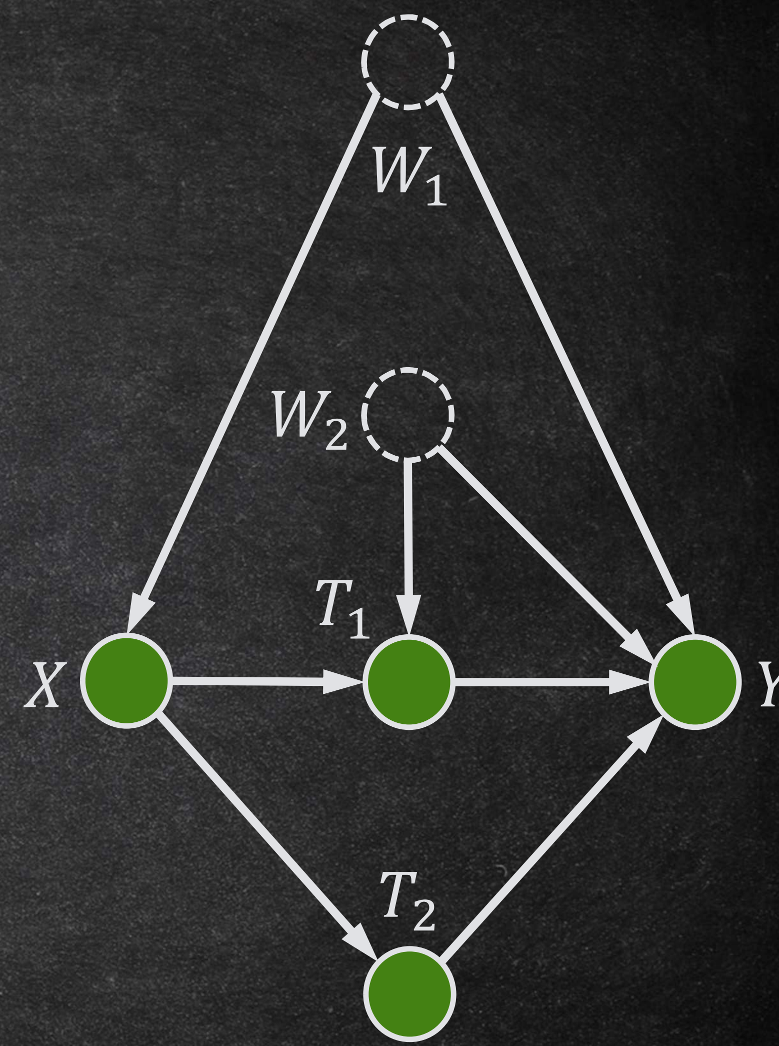


Figure 7.5(b)

We already clarified that the **UNCONFOUNDED CHILDREN CRITERION** is not necessary for identifiability, but it might aid your graphical intuition to have a **NECESSARY CONDITION** in mind.

A concrete example is as follows:

- for each backdoor path from X to any child $T \in ch(X)$ of X that is an ancestor $an(Y)$ of Y, it is possible to block that path.

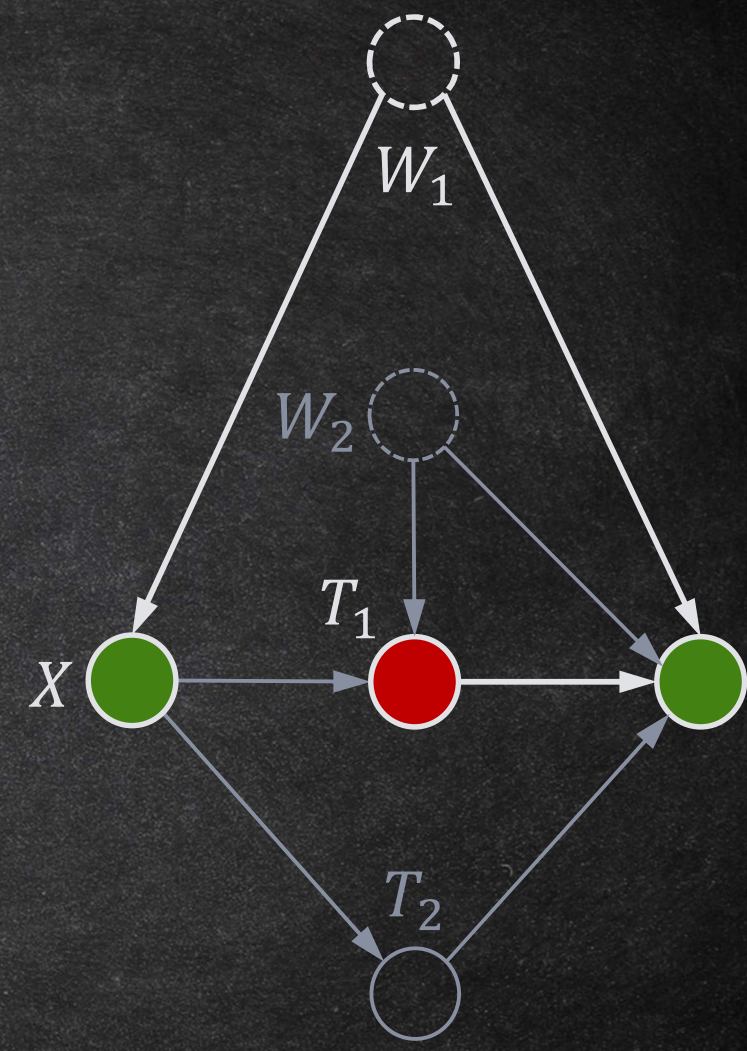


Figure 7.5(g)

We already clarified that the **UNCONFOUNDED CHILDREN CRITERION** is not necessary for identifiability, but it might aid your graphical intuition to have a **NECESSARY CONDITION** in mind.

A concrete example is as follows:

- for each backdoor path from X to any child $T \in ch(X)$ of X that is an ancestor $an(Y)$ of Y, it is possible to block that path.

  The intuition for this is that because the causal association that flows from X to Y must go through children $T \in ch(X)$ of X that are ancestors $an(Y)$ of Y, to be able to isolate this causal association, the effect of X on these mediating children $T \in ch(X)$ must be unconfounded.

  And a prerequisite to these $X - T$ (parent-child) relationships being unconfounded is that any single backdoor path from X to T must be blockable (what we state in this condition).

  Unfortunately, this condition is not sufficient.

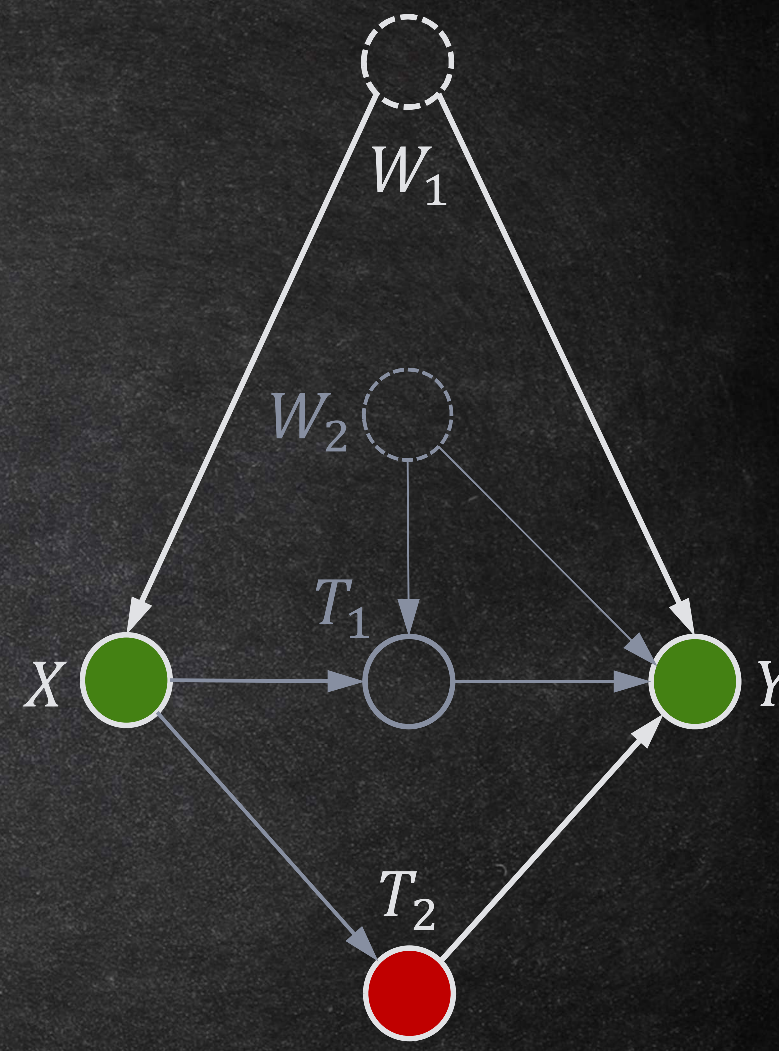
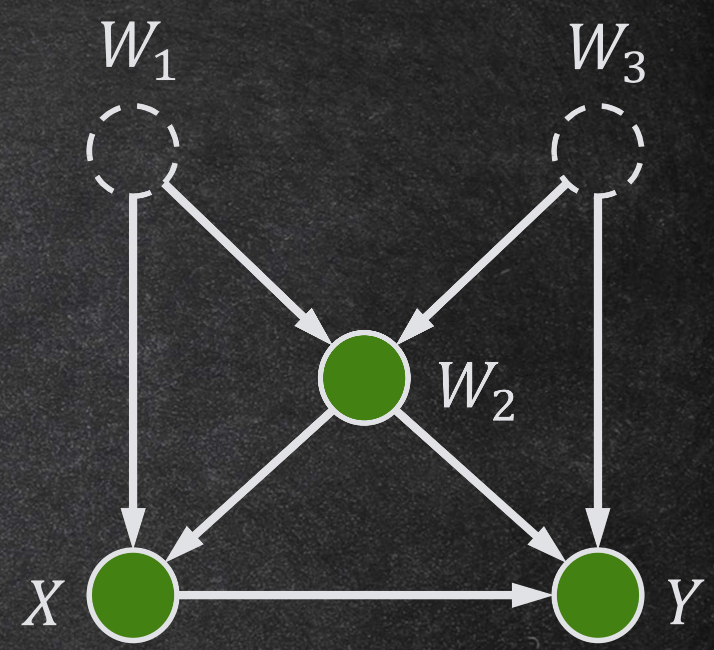


Figure 7.5(h)

The graph in Figure 7.6 has the following backdoor paths

Figure 7.6



$X \leftarrow W_1 \rightarrow W_2 \leftarrow W_3 \rightarrow Y$

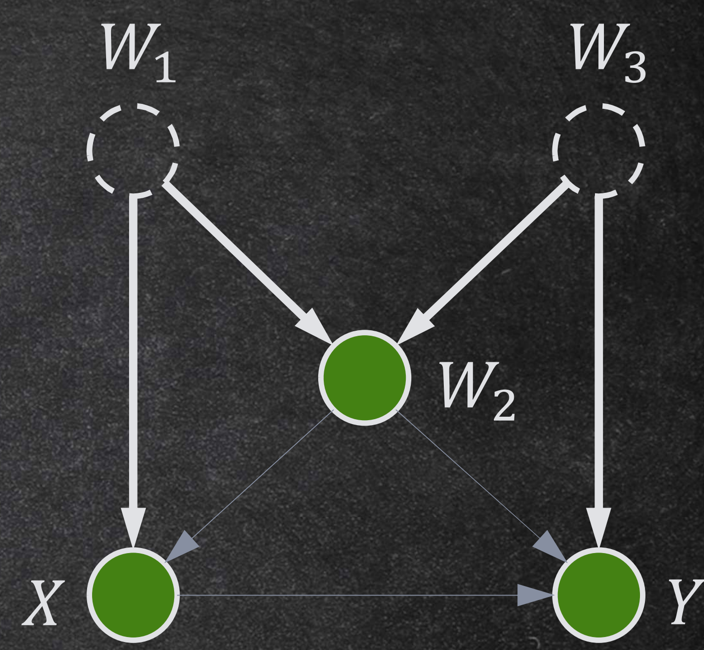
$X \leftarrow W_2 \rightarrow Y$

The graph in Figure 7.6 has the following backdoor paths

$X \leftarrow W_1 \rightarrow W_2 \leftarrow W_3 \rightarrow Y$ ⇐ blocked by collider W_2 when not conditioned on

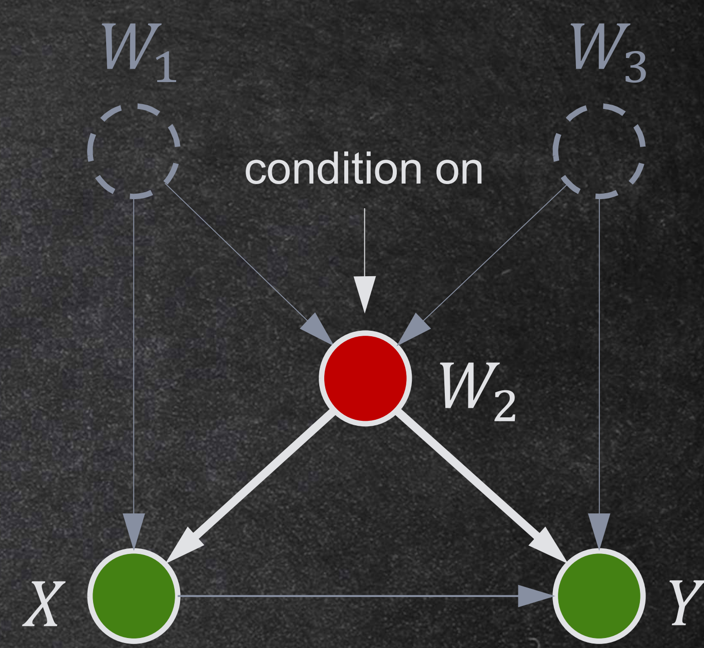
$X \leftarrow W_2 \rightarrow Y$

Figure 7.6



The graph in Figure 7.6 has the following backdoor paths

Figure 7.6



$X \leftarrow W_1 \rightarrow W_2 \leftarrow W_3 \rightarrow Y$

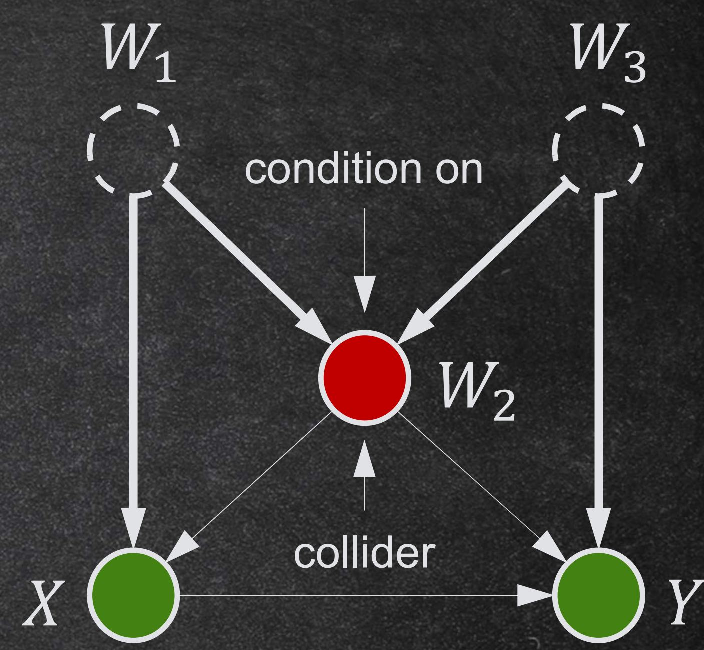
$X \leftarrow W_2 \rightarrow Y$ ⇐ blocked by conditioning on W_2 ⇒ W_2 is a collider

The graph in Figure 7.6 has the following backdoor paths

$X \leftarrow W_1 \rightarrow W_2 \leftarrow W_3 \rightarrow Y$ ⇐ conditioning on W_2 unblocks the backdoor path

$X \leftarrow W_2 \rightarrow Y$ ⇐ blocked by conditioning on W_2 ⇒ W_2 is a collider ⇑

Figure 7.6



Being able to block both paths individually does not mean we can block them both with a single conditioning set $\boldsymbol{S}$.

In conclusion, the **UNCONFOUNDED CHILDREN CRITERION** is sufficient but not necessary, and this related condition is necessary but not sufficient.

Also, everything we've seen in this section so far is for a single variable intervention.

Shpitser and Pearl provide a necessary and sufficient criterion for identifiability of

$$P\big(\mathbf{Y} = \boldsymbol{y} | do(\mathbf{X} = \boldsymbol{x})\big)$$

when $\mathbf{Y}$ and $\mathbf{X}$ are arbitrary sets of variables: **THE HEDGE CRITERION**.

However, this is rather technical and outside the scope of this course, as it requires more complex objects such as hedges, C-trees, and other leafy objects.

Moving further along, Shpitser and Pearl provide a necessary and sufficient criterion for the most general type of causal estimand: **CONDITIONAL CAUSAL EFFECTS**, which takes the form:

$$P(\mathbf{Y} = \boldsymbol{y} | do(\mathbf{X} = \boldsymbol{x}), \mathbf{Z} = \boldsymbol{z})$$

where $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{Z}$ are all arbitrary sets of variables.