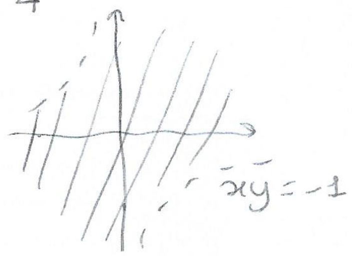
ESERCIZIO 1

$$f(x,y) = \log(1+xy) + 4$$



Dominio : $1+xy > 0 \iff xy > -1$

Curve di livello : $f(x,y) = \text{cost} \iff 1+xy = \text{cost}$

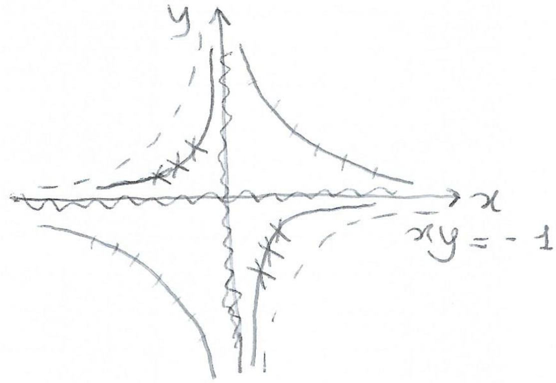
$\iff xy = \text{cost}$

$$\Rightarrow \{xy = c\}_{c>-1}$$

$c \in (-1, 0)$

$c = 0$

$c > 0$



ESERCIZIO 2

$$f(x,y) = \frac{x+y}{|x|+|y|}$$

* $\underline{x = y}$: $f(x,x) = \dfrac{2x}{2|x|} = \text{sgn}(x) \xrightarrow[x \to +\infty]{x \to 0^{\pm}} \pm 1 \Rightarrow \nexists \lim\limits_{\substack{(x,y)\to(0,0) \\ \|(x,y)\| \to +\infty}} f(x,y)$.

ESERCIZIO 3

$f_1(x,y) = e^{xy} - x\sin y \quad , \quad f_2(x,y) = xy^2 + x^2 + y + y^2$.

$$P_1^{(0,0)}(x,y) = 1 + (ye^{xy} - \sin y)\big|_0 \, x + (xe^{xy} - x\cos y)\big|_0 \, y +$$

$$+ \frac{1}{2}(x,y)\begin{bmatrix} y^2 e^{xy} & e^{xy} + xye^{xy} - \cos y \\ e^{xy} + xye^{xy} - \cos y & x^2 e^{xy} + x\sin y \end{bmatrix}\Bigg|_0 \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

$$P_1^{(2,1)}(x,y) = e^2 - 2\sin 1 + (e^2 - \sin 1)(x-2) + (2e^2 - 2\cos 1)(y-1) +$$

$$+ \frac{1}{2}(x-2, y-1)\begin{bmatrix} e^2 & 3e^2 - \cos 1 \\ 3e^2 - \cos 1 & 4e^2 + 2\sin 1 \end{bmatrix}\begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = \dots$$

$$P_2^{(0,0)}(x,y) = (y^2 + 2x)\big|_0 \, x + (2xy + 1 + 2y)\big|_0 \, y + \frac{1}{2}(x,y)\begin{bmatrix} 2 & 2y \\ 2y & 2x+2 \end{bmatrix}\Bigg|_0 \begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= y + x^2 + y^2$$

$$P_2^{(2,1)}(x,y) = 8 + 5(x-2) + 7(y-1) + \frac{1}{2}(x-2, y-1)\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = \dots$$

$P := (0,0,1,1)$

$\boxed{\text{ESERCIZIO 4}}$ $\pi : \quad t-1 = \left.\frac{\partial f}{\partial x}\right|_P x + \left.\frac{\partial f}{\partial y}\right|_P y + \left.\frac{\partial f}{\partial z}\right|_P (z-1) =$

$= 2(z-1) = 2z-2$

$\Rightarrow \pi : \quad t = 2z-1.$

$\boxed{\text{ESERCIZIO 5}}$ $\alpha > 0$

$$\lim_{(x,y)\to(1,0)} \frac{|x-1+y|^\alpha}{e^{\sqrt{(x-1)^2+y^2}}-1} = \lim_{(x,y)\to(0,0)} \frac{|x+y|^\alpha}{e^{\sqrt{x^2+y^2}}-1} = $$

COORDINATE POLARI

$$= \lim_{r\to 0} \frac{|r\cos\theta + r\sin\theta|^\alpha}{e^r-1} = \lim_{r\to 0} r^{\alpha-1} |\cos\theta+\sin\theta|^\alpha = (*)$$

▶ $\underline{\alpha > 1}$: $(*) \leq \lim_{r\to 0} r^{\alpha-1}\, 2^\alpha = 2^\alpha \cdot \lim_{r\to 0} r^{\alpha-1} = 0 \Rightarrow$ il limite $\exists$ ed è nullo.

▶ $\underline{\alpha = 1}$: $(*) = \lim_{r\to 0} 1\, |\cos\theta+\sin\theta|^\alpha = |\cos\theta+\sin\theta|^\alpha$

$\Rightarrow$ il limite $\nexists$, perché dipende da θ.
per esempio: per $\theta = 0$ si ha 1, per $\theta = \frac{3}{4}\pi$ si ha 0.

▶ $\underline{\alpha < 1}$: $(*) = \lim_{r\to 0} r^{\alpha-1} |\cos\theta+\sin\theta|^\alpha$

Nemmeno qui il limite $\exists$, per esempio:
per $\theta = 0$ si ha $+\infty$, per $\theta = \frac{3}{4}\pi$ si ha 0.

$\boxed{\text{ESERCIZIO 6}}$ $f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{y^4+4x^4}} & \text{se } (x,y)\neq(0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$

$\alpha, \beta > 0$

(a) $$\frac{|x|^\alpha |y|^\beta}{\sqrt{y^4+4x^4}} \leq \frac{(x^4)^{\alpha/4} (y^4)^{\beta/4}}{\sqrt{x^4+y^4}} \leq (x^4+y^4)^{\frac{\alpha}{4}+\frac{\beta}{4}-\frac{1}{2}} = (*)$$

$(x^4+4y^4 \geq x^4+y^4)$

$\left(a^\alpha b^\beta \leq (a+b)^{\alpha+\beta} \text{ per } a,b>0 \text{ e } \alpha,\beta>0.\right)$

▶ $\underline{\alpha+\beta > 2}$: $\lim_{(x,y)\to(0,0)} (*) = 0 \Rightarrow f$ è continua in $(0,0)$.

▶ $\underline{\alpha+\beta = 2}$: considero $y = 0 \Rightarrow f(x,0) = 0 \xrightarrow{x\to 0} 0$
considero $y = x \Rightarrow f(x,x) = \frac{1}{\sqrt{5}} \xrightarrow{x\to 0} \frac{1}{\sqrt{5}} \neq 0$

$\Rightarrow f$ <u>non</u> è continua in $(0,0)$.

▶ $\underline{\alpha+\beta < 2}$: cons. $y = 0 \Rightarrow f(x,0) = 0 \xrightarrow{x\to 0} 0$.

cons. $y = x \Rightarrow f(x,x) = \frac{1}{\sqrt{5}} |x|^{\alpha+\beta-2} \xrightarrow{x\to 0} +\infty$

$\Rightarrow f$ <u>non</u> è continua in $(0,0)$.

(b) Siano $v_1, v_2 \in \mathbb{R}$ t.c. $v_1^2 + v_2^2 = 1$.

$$\frac{f(tv_1, tv_2) - \cancel{f(0,0)}}{t} = \frac{|tv_1|^\alpha |tv_2|^\beta}{t\sqrt{4(tv_1)^4 + (tv_2)^4}} = \frac{|t|^{\alpha+\beta}}{t \cdot |t|^2} \cdot \underbrace{\frac{|v_1|^\alpha |v_2|^\beta}{\sqrt{4v_1^4 + v_2^4}}}_{=:\gamma} =$$

$$= |t|^{\alpha+\beta-3} \operatorname{sgn}(t) \cdot \gamma \quad =(*)$$

- $\underline{\alpha+\beta>3}$: $\exists \lim_{t\to 0} (*) = 0 \Rightarrow \exists$ der. diret. di f in $(0,0)$ e sono tutte nulle.
- $\underline{\alpha+\beta=3}$: $\nexists \lim_{t\to 0^\pm} (*) = \pm\gamma$
- $\underline{\alpha+\beta<3}$: $\nexists \lim_{t\to 0^+} (*) = \pm\infty$

$\Rightarrow \nexists$ der. diret. di f in $(0,0)$.

(c) Studio solo il caso $\alpha+\beta>3$, perché per $\alpha+\beta\le 3$ $\nexists$ der. direzionali.

$$\frac{f(x,y) - \cancel{f(0,0)} - \frac{\partial f}{\partial x}\cancel{(0,0)}\cdot x - \frac{\partial f}{\partial y}\cancel{(0,0)}\cdot y}{\sqrt{x^2+y^2}} =$$

$$= \frac{|x|^\alpha |y|^\beta}{\sqrt{y^4+4x^4}} \cdot \frac{1}{\sqrt{x^2+y^2}} \le \frac{|x|^{\alpha-\frac{1}{2}} |y|^{\beta-\frac{1}{2}}}{\sqrt{x^4+y^4}} \frac{|x|^{1/2}|y|^{1/2}}{\sqrt{x^2+y^2}} \le$$

$$\frac{|x|^{1/2}|y|^{1/2}}{\sqrt{x^2+y^2}} \le \frac{1}{\sqrt{2}} \le 1$$

$$\le \frac{(x^4)^{\frac{\alpha-1/2}{4}} (y^4)^{\frac{\beta-1/2}{4}}}{\sqrt{x^4+y^4}} \le (x^4+y^4)^{\frac{\alpha-1/2}{4}+\frac{\beta-1/2}{4}-\frac{1}{2}} =$$

$$= (x^4+y^4)^{\frac{2\alpha-1+2\beta-1-4}{8}} = (x^4+y^4)^{\frac{2\alpha+2\beta-6}{8}} =$$

$$= (x^4+y^4)^{\frac{\alpha+\beta-3}{4}} \xrightarrow[(x,y)\to(0,0)]{} 0 \quad \text{per } \alpha+\beta>3.$$

Quindi f è differenziabile nell'origine.

$\boxed{\text{ESERCIZIO 7}}$ $f(x,y) = \left(x^2 e^{x+y}, \int_{xy}^{1} e^{t^6}\,dt\right)$

$$\mathrm{Jac}_{(x,y)}(f) = \begin{bmatrix} 2xe^{x+y} + x^2e^{x+y} & x^2e^{x+y} \\ -ye^{x^6y^6} & -xe^{x^6y^6} \end{bmatrix}$$

$$\mathrm{Jac}_{(2,0)}(f) = \begin{bmatrix} 4e^2 + 4e^2 & 4e^2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 8e^2 & 4e^2 \\ 0 & -2 \end{bmatrix}$$

ESERCIZIO 8 $\quad f(x,y) = \begin{cases} \sqrt[3]{y}\, e^{-\frac{y^2}{x^4}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

(a) f è continua sicuramente in $\mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 \mid x\neq 0\}$.

▶ $(0,y)$ con $y\neq 0$:

$$f(0+h, y+k) - f(0,\cancel{y}) = \underbrace{(y+k)^{1/3}}_{\xrightarrow{k\to 0} y^{1/3}\in\mathbb{R}} \underbrace{e^{-\frac{(y+k)^2}{h^4}}}_{\to 0} \xrightarrow{(h,k)\to(0,0)} 0$$

$(y+k)^2 \to y^2 \neq 0$, $-\frac{(y+k)^2}{h^4} \to -\infty$

▶ $(0,0)$: $\quad f(h,k) - f(\cancel{0,0}) = k^{1/3} \underbrace{e^{-\frac{k^2}{h^4}}}_{\leq 1} \leq k^{1/3} \xrightarrow{k\to 0} 0$

$\Rightarrow f$ è continua in $\mathbb{R}^2$.

(b) $\exists$ gradiente $\Leftrightarrow$ $\exists$ derivate parziali $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Sicuramente, $\exists$ gradiente in (x,y) per ogni

$$(x,y) \in \mathbb{R}^2 \setminus \{xy = 0\} = \mathbb{R}^2 \setminus \{\text{assi } x, y\}.$$

▶ $(0,y)$ (ASSE y): Essendo $f \equiv 0$ sull'asse y, $\exists \frac{\partial f}{\partial y}(0,y) = 0$ $\forall y\in\mathbb{R}$

$$\frac{f(h,y) - \cancel{f(0,y)}}{h} = \frac{y^{1/3}}{h} e^{-\frac{y^2}{h^4}} \xrightarrow{h\to 0} 0 \Rightarrow \exists \frac{\partial f}{\partial x}(0,y) = 0 \quad \forall y\in\mathbb{R}.$$

In particolare, $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

▶ $(x,0)$, $x\neq 0$ (ASSE x, TRANNE L'ORIGINE): Essendo $f\equiv 0$ sull'asse x, $\exists \frac{\partial f}{\partial x}(x,0) = 0$ $\forall x\in\mathbb{R}$.

$$\frac{f(x,k) - \cancel{f(x,0)}}{k} = \frac{k^{1/3}}{k} e^{-\frac{k^2}{x^4}} =$$

$$= k^{-2/3} e^{-\frac{k^2}{x^4}} \xrightarrow{k\to 0} +\infty.$$

Quindi il gradiente di f esiste in tutti e soli i punti di

$$\mathbb{R}^2 \setminus \{(x,0) \mid x\neq 0\}.$$

(c) Fisso $v_1, v_2 \in \mathbb{R}$ con $v_1^2 + v_2^2 = 1$.

$$\frac{f(tv_1, tv_2) - f(\cancel{0},0)}{t} = \frac{t^{1/3}\, v_2^{1/3}}{t}\, e^{-\frac{v_2^2}{t^2 v_1^4}} = (*)$$

▶ se $\underline{v_2 = 0}$: $(*) = 0 \xrightarrow{t \to 0} 0$

▶ se $\underline{v_2 \neq 0}$: $(*) = t^{-2/3}\, v_2^{1/3}\, e^{-\frac{1}{t^2}\frac{v_2^2}{v_1^4}} \xrightarrow{t \to 0} 0$

Quindi $\lim_{t \to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = 0$

$\forall\, v_1, v_2 \in \mathbb{R} :\ v_1^2 + v_2^2 = 0$ ossia

tutte le derivate direzionali nell'origine esistono e sono nulle.

(d) $$\frac{f(x,y) - \cancel{f(0,0)} - \cancel{\frac{\partial f}{\partial x}(0,0)} \cdot x - \cancel{\frac{\partial f}{\partial y}(0,0)} \cdot y}{\sqrt{x^2+y^2}} =$$

$$= \frac{\sqrt[3]{y}}{\sqrt{x^2+y^2}}\, e^{-\frac{y^2}{x^4}} = (*)$$

L'idea è "ammazzare" l'esponenziale (che era ciò che ci risolveva i problemi, mandando tutto a zero), quindi consideriamo

$$y = x^2.$$

Allora

$$(*) \overset{y=x^2}{=} \frac{x^{2/3}}{\sqrt{2}\,|x|}\, e^{-1} = \frac{e^{-1}}{\sqrt{2}}\, |x|^{-1/3} \xrightarrow[x \to 0]{} +\infty .$$

Quindi f <u>non</u> è differenziabile in $(0,0)$.