FOGLIO 5

1. CAMBIO DI COORDINATE : $\begin{cases} u := yx^2 \\ v := \frac{y}{x} \end{cases} \longleftrightarrow \begin{cases} x = \sqrt[4]{\frac{u}{v}} \\ y = \sqrt[3]{uv^2} \end{cases}$

$\begin{pmatrix} uv^2 = y^3 \\ \frac{u}{v} = x^4 \end{pmatrix}$

$$\int_D \frac{y}{1+yx^2}\,dx\,dy = \int_1^2\int_1^2 \frac{\cancel{\sqrt[3]{uv^2}}}{1+u}\,\frac{1}{3\cancel{\sqrt[3]{uv^2}}}\,du\,dv =$$

$$\psi(x,y) := \begin{pmatrix} yx^2 \\ \frac{y}{x} \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$$\det\left(Jac_{(x,y)}\psi\right) = \det\begin{bmatrix} 2xy & x^2 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix} = 2y + y = 3y$$

$$\det\left(Jac_{(u,v)}\psi^{-1}\right) = \frac{1}{\det\left(Jac_{\psi^{-1}(u,v)}\psi\right)} = \frac{1}{3\sqrt[3]{uv^2}}$$

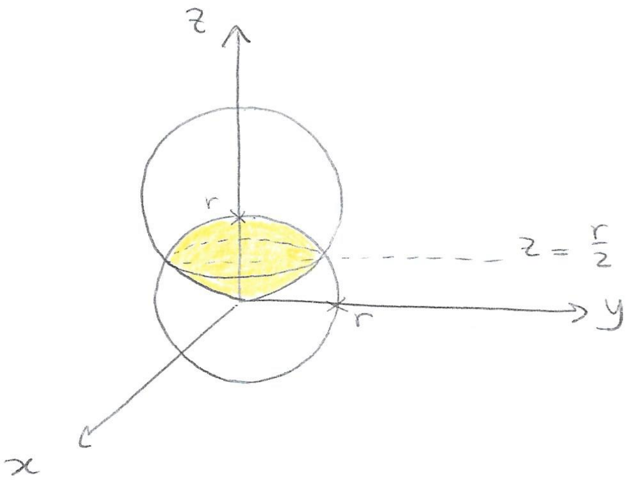
NOTA: CONSIDERO det(Jac(...)) SENZA MODULO, PERCHE' RISULTA "3y" E $y \in [1,2]$ QUINDI $|3y| = 3y$.

$$= \frac{1}{3}\int_1^2 dv \int_1^2 \frac{1}{1+u}\,du = \frac{1}{3}\ln(1+u)\Big|_1^2 = \frac{1}{3}\ln\left(\frac{3}{2}\right) \blacksquare$$

② $A_r = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le r^2,\ x^2+y^2+(z-r)^2\le r^2\}$

(a) $\mathrm{vol}(A_r) \underset{\uparrow\ \text{PER SIMMETRIA}}{=} 2\,\mathrm{vol}(A_r^+) =$

$\left(A_r^+ := \{(x,y,z)\in A_r \mid z \ge \frac{r}{2}\}\right)$



$$= 2\int_{\frac{r}{2}}^{r}\int_0^{2\pi}\int_0^{\sqrt{r^2-z^2}} \rho\, d\rho\, d\theta\, dz =$$

$$= \cancel{2}\cdot 2\pi \int_{\frac{r}{2}}^{r} \frac{r^2-z^2}{\cancel{2}}\, dz =$$

$$= 2\pi\left[r^2\left(r-\frac{r}{2}\right) - \frac{1}{3}\left(r^3-\frac{r^3}{8}\right)\right] = 2\pi r^3\left[\frac{1}{2}-\frac{1}{3}\cdot\frac{7}{8}\right] =$$

$$= 2\pi r^3\,\frac{12-7}{3\cdot 8} = \frac{5}{12}\pi r^3 \ ✓$$

(b) $$\int_{A_r} z^2\, dx\,dy\,dz = \int_{\frac{r}{2}}^{r}\int_0^{2\pi}\int_0^{\sqrt{r^2-z^2}} \rho z^2\, d\rho\, d\theta\, dz + \int_0^{\frac{r}{2}}\int_0^{2\pi}\int_0^{\sqrt{2zr-z^2}} \rho z^2\, d\rho\, d\theta\, dz$$

$$= \cancel{2}\pi\int_{\frac{r}{2}}^{r} \frac{r^2-z^2}{\cancel{2}}\, z^2\, dz + \cancel{2}\pi\int_0^{\frac{r}{2}} \frac{2zr-z^2}{\cancel{2}}\, z^2\, dz =$$

$$= \pi\left[r^2\cdot\frac{1}{3}\left(r^3-\frac{r^3}{8}\right) - \frac{1}{5}\left(r^5-\cancel{\frac{r^5}{32}}\right) + 2r\cdot\frac{1}{4}\left(\frac{r^4}{16}-0\right) - \cancel{\frac{1}{5}\left(\frac{r^5}{32}-0\right)}\right] =$$

$$= \pi r^5\left[\frac{1}{3}\cdot\frac{7}{8} - \frac{1}{5} + \frac{1}{32}\right] = \frac{59}{480}\pi r^5 \ ✓$$

(C) PER $\alpha \geq 0$, SICURAMENTE QUELL'INTEGRALE ESISTE FINITO, ESSENDO L'INTEGRANDA LIMITATA E IL DOMINIO COMPATTO.

STUDIO IL CASO $\alpha < 0$.

PER SIMMETRIA, $\int_{A_2} (y^2+(z-1)^2+x^2)^{\alpha}\, dx\,dy\,dz = 2\int_{A_2^+} (y^2+(z-1)^2+x^2)^{\alpha}\, dx\,dy\,dz$

DOVE

$$\int_{A_2^+} (y^2+(z-1)^2+x^2)^{\alpha}\, dx\,dy\,dz = \leftarrow \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ \xi = z-1 \end{cases} \quad \begin{array}{l} \theta \in [0, 2\pi] \\ z \in [1,2] \leftrightarrow \xi \in [0,1] \\ \rho \in [0, \sqrt{4-z^2}] = \\ = [0, \sqrt{4-(\xi+1)^2}] \end{array}$$

$$= \frac{1}{2}\int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-(\xi+1)^2}} (\rho^2+\xi^2)^{\alpha}\, 2\rho\, d\rho\, d\theta\, d\xi \underset{\substack{\uparrow \\ \text{CONS. } \alpha \neq -1}}{=} (*)$$

DERIVATA DI ρ^2 (RISPETTO A ρ)

$$= \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \int_0^1 \left[\frac{(\rho^2+\xi^2)^{\alpha+1}}{\alpha+1}\right]_0^{\sqrt{4-(\xi+1)^2}} d\xi =$$

$$= \frac{\pi}{\alpha+1}\int_0^1 \left[(4-\cancel{\xi^2}-2\xi-1+\cancel{\xi^2})^{\alpha+1} - \xi^{2(\alpha+1)}\right] d\xi =$$

$$= \underbrace{\frac{\pi}{\alpha+1}\int_0^1 (3-2\xi)^{\alpha+1}\, d\xi}_{<\infty \ \forall \alpha} - \underbrace{\frac{\pi}{\alpha+1}\int_0^1 \frac{1}{\xi^{-2\alpha-2}}\, d\xi}_{<\infty}$$

SSE $-2\alpha-2 < 1$

SSE $2\alpha > -3$

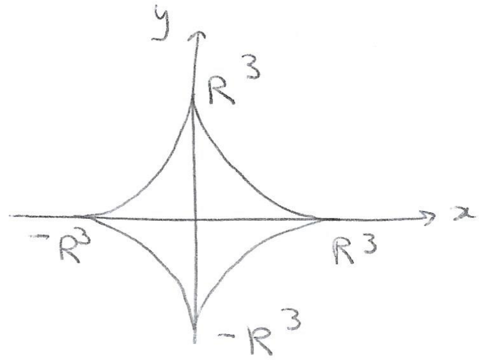
SSE $\alpha > -\frac{3}{2}$. $(\alpha \neq -1)$.

SIA ORA $\alpha = -1$. ALLORA OTTENGO

$$(*) = \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \int_0^1 \left[\ln(\rho^2+\xi^2)\right]_0^{\sqrt{4-(\xi+1)^2}} d\xi = \underbrace{\pi\int_0^1 \ln(3-2\xi)\, d\xi}_{<\infty} - \pi\underbrace{\int_0^1 \ln \xi^2\, d\xi}_{= 2\int_0^1 \ln \xi\, d\xi < \infty}$$

QUINDI $\int_{A_2} (y^2+(z-1)^2+x^2)\, dx\,dy\,dz < \infty \leftrightarrow \alpha > -\frac{3}{2}$.

③ $D_R = \{(x,y)\in\mathbb{R}^2 \mid \sqrt[3]{|x|}+\sqrt[3]{|y|} \le R\}$



(a) $e^{-|x|-|y|} \ge 0 \qquad \forall (x,y)\in\mathbb{R}^2$

$$\Rightarrow \lim_{R\to+\infty}\int_{D_R} e^{-|x|-|y|}\,dx\,dy = \int_{\mathbb{R}^2} e^{-|x|-|y|}\,dx\,dy =$$

$$= 4\lim_{R\to+\infty}\int_0^R\int_0^R e^{-x-y}\,dx\,dy =$$

$$= 4\lim_{R\to+\infty}\left(\int_0^R e^{-x}\,dx\right)\left(\int_0^R e^{-y}\,dy\right) =$$

$$= 4\lim_{R\to+\infty}(1-e^{-R})^2 = 4\cdot 1 = 4 \quad ✓$$

(b) OSSERVO CHE LUNGO $x+y=0$ SI HA $e^{-|x+y|}\equiv 1$.

IDEA: L'INTEGRALE DI UNA COSTANTE >0 (PER ESEMPIO "1") SU UN DOMINIO ILLIMITATO (PER ESEMPIO $\{(x,y)\in\mathbb{R}^2 \mid x+y=0\}$) È $=+\infty$. QUESTO VALE SE L'INTEGRALE È SU UN INSIEME DI MISURA NON NULLA, QUINDI NEL NOSTRO CASO

$$\int_{\{(x,y)\in\mathbb{R}^2 \mid x+y=0\}} 1\,dx = 0 \qquad \text{(INT. IN UNA VARIABILE)}$$

MA

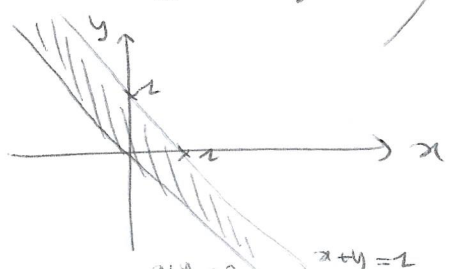
$$\int_{\{(x,y)\in\mathbb{R}^2 \mid x+y=0\}} 1\,dx\,dy = 0 \qquad \text{(INT. IN 2 VARIABILI)}$$

1 DIMENSIONE!

QUINDI, SE VOGLIO SFRUTTARE L'OSSERVAZIONE CHE $e^{-|x+y|}=1$ LUNGO $\{x+y=0\}$ ALLORA DEVO ESTENDERE IL DOMINIO DI INTEGRAZIONE AD UN INSIEME DI MISURA NON NULLA.

CONSIDERO DUNQUE $\{(x,y)\in\mathbb{R}^2 \mid 0\le x+y\le 1\}$.

$P := \{(x,y)\in\mathbb{R}^2 \mid 0 \le x+y \le 1\}$



$$\lim_{R\to+\infty}\int_{D_R}\underbrace{e^{-|x+y|}}_{\geq 0}\,dx\,dy = \int_{\mathbb{R}^2}\underbrace{e^{-|x+y|}}_{\geq 0}\,dx\,dy \geq \int_P e^{-|x+y|}\,dx\,dy =$$

$$= \int_P e^{-x-y}\,dx\,dy = \int_{-\infty}^{+\infty} e^{-x}\left(\int_{-x}^{-x+1} e^{-y}\,dy\right)dx =$$

$$= -\int_{-\infty}^{+\infty} e^{-x}\left[e^{-(-x+1)} - e^{-(-x)}\right]dx = \int_{-\infty}^{+\infty} e^{-x}\left[e^{x} - e^{x-1}\right]dx =$$

$$= \int_{-\infty}^{+\infty}\underbrace{(1-e^{-1})}_{>0}\,dx = +\infty.$$

QUINDI $\displaystyle\lim_{R\to+\infty}\int_{D_R} e^{-|x+y|}\,dx\,dy = +\infty.$ ✓

(c) $\forall R>0$ D_R è un dominio simmetrico rispetto alla x, mentre $g(x) := x e^{-|x+y|}$ è una funzione dispari. ($\forall y \in \mathbb{R}$)

Dunque $\forall R>0$ $\displaystyle\int_{D_R} x e^{-|x+y|}\,dx\,dy = 0$

$$\Rightarrow \underbrace{\lim_{R\to+\infty}\int_{D_R} x e^{-|x+y|}\,dx\,dy}_{} = 0.$$

$\left(\neq \displaystyle\int_{\mathbb{R}^2} x e^{-|x+y|}\,dx\,dy\right.$, che non esiste.$\Big)$

4 (a) COORDINATE CILINDRICHE: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$

$$(x,y,z)\in D \iff \begin{cases} \rho^2\cos^2\theta < z^2 < \rho^2 \\ \theta \in (0, \pi/2) \\ 0 < z < \rho^2 < 1 \end{cases} \qquad \iff \rho\cos\theta < z < \rho$$

$$(\Rightarrow 0 < z < 1,\ 0 < \rho < 1)$$

dove $\begin{cases} 0 < z < \rho^2 \\ \rho\cos\theta < z < \rho \end{cases} \iff \rho\cos\theta < z < \rho^2 \quad (0 < \rho < 1).$

$$(\Rightarrow \cos\theta < \rho)$$

QUINDI

$$\int_D f\, dx\,dy\,dz = \int_0^{\pi/2}\int_{\cos\theta}^{1}\int_{\rho\cos\theta}^{\rho^2} \frac{\rho\sin\theta}{(\rho^2+z^2)\cancel{\rho}}\,\cancel{\rho}\, dz\, d\rho\, d\theta =$$

$$= \int_0^{\pi/2}\int_{\cos\theta}^{1}\int_{\rho\cos\theta}^{\rho^2} \frac{1}{\cancel{\rho}}\,\frac{\frac{1}{\rho}}{\left(1+\left(\frac{z}{\rho}\right)^2\right)}\, dz\, \cancel{\rho}\, d\rho\, \sin\theta\, d\theta =$$

$$= \int_0^{\pi/2}\int_{\cos\theta}^{1} \left[\operatorname{artg}\left(\frac{z}{\rho}\right)\right]\Bigg|_{z=\rho\cos\theta}^{z=\rho^2} d\rho\, \sin\theta\, d\theta =$$

$$= \int_0^{\pi/2}\int_{\cos\theta}^{1} \left[\operatorname{artg}(\rho) - \operatorname{artg}(\cos\theta)\right] d\rho\, \sin\theta\, d\theta =$$

$$\left[\int 1\cdot\operatorname{artg}(\rho)\,d\rho \overset{P.P.}{=} \rho\operatorname{artg}(\rho) - \frac{1}{2}\int \frac{2\rho}{1+\rho^2}\,d\rho = \rho\cdot\operatorname{artg}(\rho) - \frac{1}{2}\ln(1+\rho^2) + c\right]$$

$$= \int_0^{\pi/2} \left[\rho\cdot\operatorname{artg}(\rho) - \frac{1}{2}\ln(1+\rho^2) - \rho\operatorname{artg}(\cos\theta)\right]\Bigg|_{\cos\theta}^{1} \sin\theta\, d\theta =$$

$$= \int_0^{\pi/2} \left[\frac{\pi}{4} - \frac{1}{2}\ln 2 - \operatorname{artg}(\cos\theta) \cancel{-(\cos\theta)\operatorname{artg}(\cos\theta)} + \frac{1}{2}\ln(1+\cos^2\theta) \cancel{-(\cos\theta)\operatorname{artg}(\cos\theta)}\right]\sin\theta\, d\theta =$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\int_0^{\pi/2} \sin\theta\, d\theta + \int_0^1 \left[-\operatorname{artg}(t) + \frac{1}{2}\ln(1+t^2)\right] dt =$$

$$\left(\begin{cases} t := \cos\theta \\ dt = -\sin\theta\, d\theta \end{cases} \quad 0 \le \theta \le \frac{\pi}{2} \to 0 \le t = \cos\theta \le 1\right)$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\left[-\cos\theta\right]\Big|_0^{\pi/2} + \left[-t\cdot\operatorname{artg}(t) + \frac{1}{2}\ln(1+t^2) + \frac{1}{2}t\ln(1+t^2) - t + \operatorname{artg}(t)\right]\Big|_0^1 =$$

$$\left[\int 1\cdot\ln(1+t^2)\,dt \overset{P.P.}{=} t\cdot\ln(1+t^2) - \int \frac{2t^2}{1+t^2}\,dt \overset{+c}{=} t\cdot\ln(1+t^2) - 2\left[\int \frac{1+t^2}{1+t^2}\,dt - \int\frac{1}{1+t^2}\,dt\right] + c = \right.$$

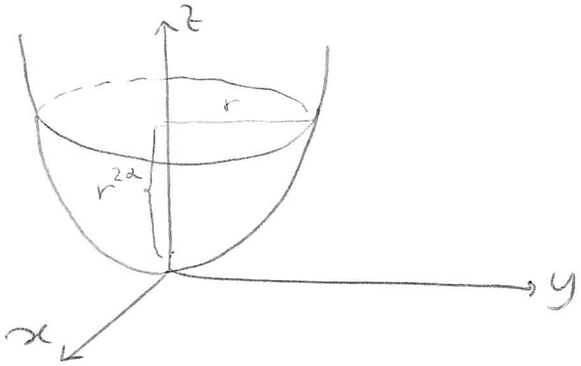
$$\left. = t\cdot\ln(1+t^2) - 2t + 2\operatorname{artg}(t) + c\right]$$

$$=\left(\cancel{\frac{\pi}{4}}-\cancel{\frac{1}{2}\ln 2}\right)[1-0]+\left[-\cancel{\frac{\pi}{4}}+\cancel{\frac{1}{2}\ln 2}+\frac{1}{2}\ln 2-1+\frac{\pi}{4}\right]=\frac{\pi}{4}+\frac{1}{2}\ln 2-1$$

QUINDI ESISTE FINITO. ✓

(b) Per simmetria:

$$\int_{E_\alpha} g=\int_{E_\alpha^+} g$$



per $E_\alpha^+ := \{(x,y,z)\in E_\alpha \mid y>0\}$

PER UNA CERTA COSTANTE $k=k(\alpha)$ → k

$$\int_{E_\alpha^+} g\,dx\,dy\,dz=\frac{1}{2}\int_0^{k}\int_0^{\pi/2}\int_0^{r^{2\alpha}}\frac{2zr\sin\theta}{(r^2+z^2)\cancel{r}}\cdot\cancel{r}\;dz\,d\theta\,dr=$$

$$=\underbrace{\frac{1}{2}\int_0^{\pi/2}\sin\theta\,d\theta}_{\text{costante}}\cdot\int_0^{k} r\left[\ln(r^2+z^2)\right]_0^{r^{2\alpha}}dr=$$

$$=C_1\int_0^{k} r\left[\ln(r^2+r^{4\alpha})-\ln r^2\right]dr=C_1\int_0^{k} r\cdot\ln\left(\frac{r^2+r^{4\alpha}}{r^2}\right)dr=$$

$$=C_1\int_0^{k} r\cdot\ln\left(1+r^{4\alpha-2}\right)dr<\infty\qquad\forall\alpha>0$$

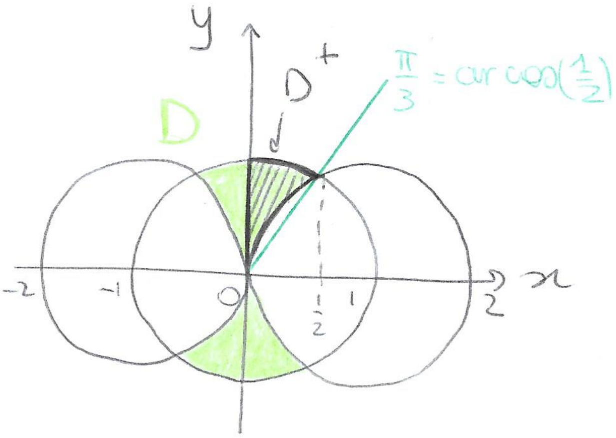
Quando tale integrale esiste, è nullo: $\int_{E_\alpha} g=0$.

(INFATTI E_α è un dominio simmetrico rispetto a y, ma $g(x,-y,z)=-g(x,y,z)$ DISPARI.)

⑤ (a)

$$\text{Area}(D) = 4\,\text{Area}(D^+) =$$

$$= 4\int_{\frac{\pi}{3}}^{\pi/2}\int_{2\cos\theta}^{1} \rho \; d\rho\, d\theta =$$



COORDINATE POLARI $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

$(x,y)\in D^+ \iff \rho^2 \le 1,\ \underbrace{(\rho\cos\theta - 1)^2 + \rho^2\sin^2\theta \ge 1}_{}$, $\sin\theta \ge 0$

SSE $\rho^2 - 2\rho\cos\theta \ge 0$

SSE $\rho \ge 2\cos\theta$

$$= 4\int_{\pi/3}^{\pi/2} \frac{1-4\cos^2\theta}{2}\, d\theta = 2\left(\frac{\pi}{2}-\frac{\pi}{3}\right) - 2\cdot 4\int_{\frac{\pi}{3}}^{\pi/2}\cos^2\theta\, d\theta =$$

$$\left(\int \overset{f}{\cos\theta}\cdot\overset{g'}{\cos\theta}\, d\theta \overset{P.P.}{=} \cos\theta\cdot\sin\theta + \int \sin\theta\cdot\sin\theta\, d\theta = \right.$$

$$= \cos\theta\cdot\sin\theta + \int (1-\cos^2\theta)\, d\theta + C$$

$$\left. 2\int \cos\theta\cdot\cos\theta\, d\theta = \cos\theta\cdot\sin\theta + \theta + C \right)$$

$$= 2\cdot\frac{\pi}{6} - 8\cdot\frac{1}{2}\Big[\cos\theta\cdot\sin\theta + \theta\Big]\Big|_{\pi/3}^{\pi/2} =$$

$$= \frac{\pi}{3} - 4\left[\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right] = \frac{\pi}{3} - 4\cdot\frac{\pi}{6} + \frac{4\sqrt{3}}{4} = \sqrt{3} - \frac{\pi}{3} \checkmark$$

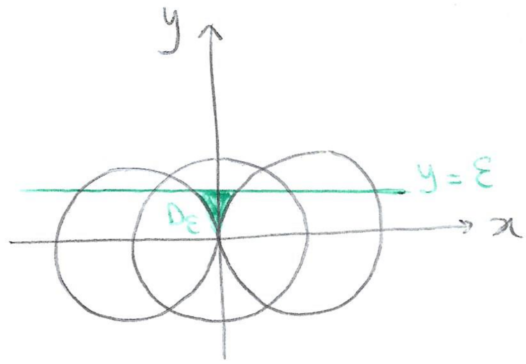
(b) $\displaystyle\int_D \frac{e^x}{|y|^\alpha \sin y}\, dx\, dy < \infty \iff \int_{D^+} \frac{e^x}{|y|^\alpha \sin y}\, dx\, dy < \infty$

PER $D^+ := \{(x,y)\in D \mid y \ge 0\}$.

In particolare, essendo l'integranda continua in $D^+ \setminus \{(0,0)\}$ allora $\displaystyle\int_D \frac{e^x}{|y|^\alpha \sin y}\, dx\, dy < \infty \iff \int_{D_\varepsilon} \frac{e^x}{|y|^\alpha \sin y}\, dx\, dy < \infty$

per $D_\varepsilon := \{(x,y)\in D \mid 0 \le y \le \varepsilon\}$, con $0 < \varepsilon < \frac{1}{2}$ (ε piccolo).

$$\int_{D_\varepsilon} \frac{e^x}{|y|^\alpha \sin y}\,dx\,dy =$$



$$= \int_0^\varepsilon \int_{-1+\sqrt{1-y^2}}^{1-\sqrt{1-y^2}} \frac{e^x}{y^\alpha \sin y}\,dx\,dy =$$

$$= \int_0^\varepsilon \underbrace{\frac{e^{1-\sqrt{1-y^2}} - e^{-1+\sqrt{1-y^2}}}{y^\alpha \sin y}}_{=:\, g(y)}\,dy$$

DOVE PER $y \sim 0$ SI HA:

$$g(y) \sim \frac{1+1-\sqrt{1-y^2} - (\cancel{1}-\cancel{1}+\sqrt{1-y^2})}{y^\alpha\, y} = 2\,\frac{1-\sqrt{1-y^2}}{y^\alpha\, y} \sim$$

($e^t \sim 1+t$ per $t \sim 0$)

$$\sim 2\,\frac{\cancel{1}-(\cancel{1}-\frac{1}{2}y^2)}{y^\alpha\, y} = \cancel{2}\,\frac{y^2}{y^\alpha\, y} = \frac{1}{y^{\alpha-1}}$$

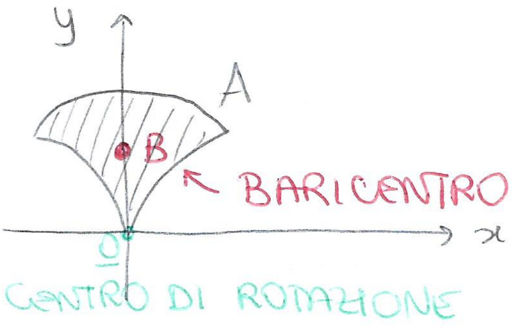
($(1+t)^a \sim 1+at$ per $t \sim 0 \Rightarrow \left(\begin{array}{l} t := -y^2 \\ a = \frac{1}{2}\end{array}\right)$)

Quindi $\int_0^\varepsilon g(y)\,dy < \infty \iff \alpha - 1 < 1 \iff \alpha < 2$.

Quando esiste, $\int_D \frac{e^x}{|y|^\alpha \sin y}\,dx\,dy = 0$

perché D è simmetrico rispetto a y
e l'integranda è dispari in y. ✓

(c) TEOREMA DI PAPPO - GULDINO:



$$A := \{(x,y) \in D \mid y > 0\}$$

$$B = (x_B, y_B) = \frac{1}{\text{area}(A)}\left(\int_A x\,dx\,dy, \int_A y\,dx\,dy\right)$$

$\text{Vol}(A_{ROTAZ}) = \leftarrow$ (TEOREMA DI PAPPO - GULDINO)

$$= 2\pi \cdot \text{dist}(B, \underline{0}) \cdot \text{area}(A)$$

DOVE $\text{dist}(B, \underline{0}) \underset{\uparrow}{=} y_B = \dfrac{1}{\text{area}(A)} \displaystyle\int_A y\,dx\,dy \quad \Rightarrow$

[PER SIMMETRIA, $B = (0, y_B)$ OSSIA $x_B = 0$]

$$\Rightarrow \text{Vol}(A_{ROTAZ}) = 2\pi \cdot \int_A y\,dx\,dy \underset{\uparrow}{=} 2\pi \cdot 2 \int_{D^+} y\,dx\,dy =$$

$$\left(D^+ = \{(x,y) \in A \mid x > 0\}\right)$$

$$= 4\pi \int_{\pi/3}^{\pi/2} \int_{2\cos\theta}^{1} \rho\sin\theta \cdot \rho\,d\rho\,d\theta =$$

$$= 4\pi \int_{\pi/3}^{\pi/2} \frac{1 - 8\cos^3\theta}{3} \sin\theta\,d\theta =$$

$$= \frac{4}{3}\pi \left[-\cos\theta + 8 \cdot \frac{\cos^4\theta}{4}\right]\Big|_{\pi/3}^{\pi/2} = \frac{4}{3}\pi \left[-\cos\theta + 2\cos^4\theta\right]\Big|_{\pi/3}^{\pi/2} =$$

$$= \frac{4}{3}\pi \left[\frac{1}{2} - 2 \cdot \frac{1}{16}\right] = \frac{4}{3}\pi \left[\frac{1}{2} - \frac{1}{8}\right] = \frac{4}{\cancel{3}}\pi \cdot \frac{\cancel{3}}{8} = \frac{\pi}{2} \quad \blacksquare$$

⑥ $E := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$

(a) $\mathrm{vol}(E) = \int_E dx\,dy\,dz \underset{\uparrow}{=} \int_{\{u^2+v^2+w^2 \leq 1\}} abc\, du\,dv\,dz =$

$$\begin{cases} u := \frac{x}{a} \\ v := \frac{y}{b} \\ w := \frac{z}{c} \end{cases}$$

$(x,y,z) \in E \Leftrightarrow u^2+v^2+w^2 \leq 1$

$$\psi(x,y,z) := \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{x}{a} \\ \frac{y}{b} \\ \frac{z}{c} \end{pmatrix} \Rightarrow \mathrm{Jac}_{(x,y,z)}\psi = \begin{bmatrix} \frac{1}{a} & & \\ & \frac{1}{b} & \\ & & \frac{1}{c} \end{bmatrix}$$

$$\det\left(\mathrm{Jac}_{(u,v,w)}\psi^{-1}\right) = \frac{1}{\det\left(\mathrm{Jac}_{\psi^{-1}(u,v,w)}\psi\right)} = abc$$

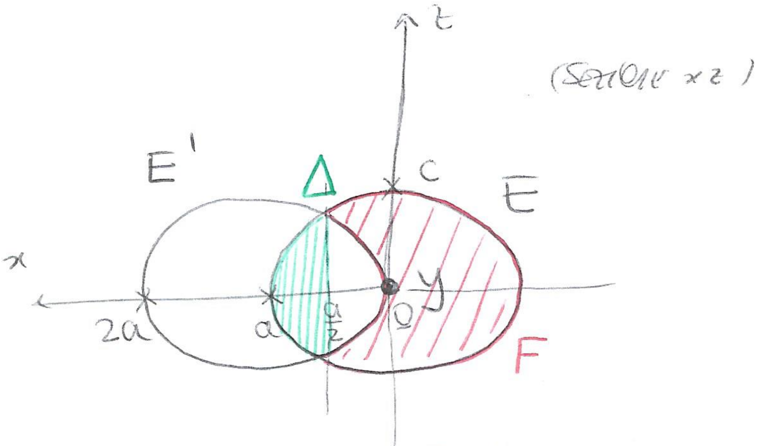
$= abc \cdot \mathrm{vol}(\text{SFERA UNITARIA 3-DIMENSIONALE}) = abc \cdot \frac{4}{3}\pi = \frac{4}{3}\pi abc$ ✓

(b) $\mathrm{vol}(\Delta) = \int_\Delta dx\,dy\,dz =$

$= \int_\Delta abc\, du\,dv\,dw =$

$\underset{\uparrow}{=} abc \int_{\frac{1}{2}}^{1} \int_0^{2\pi} \int_0^{\sqrt{1-u^2}} \rho\, d\rho\, d\theta\, du =$

$$\left(\begin{cases} u = u \\ v = \rho\cos\theta \\ w = \rho\sin\theta \end{cases} \right)$$



$= abc \cdot \cancel{2}\pi \int_{1/2}^{1} \frac{1-u^2}{\cancel{2}}\, du = \pi abc \left[u - \frac{u^3}{3} \right]\Big|_{1/2}^{1} =$

$= \pi abc \left[1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{8} \right] = \pi abc\, \frac{24-8-12+1}{3\cdot 8} = \frac{5}{24}\pi abc$

$\mathrm{vol}(F) = \mathrm{vol}(E) - 2\,\mathrm{vol}(\Delta) = \left(\frac{4}{3} - 2\cdot\frac{5}{24} \right)\pi abc =$

$= \left(\frac{4}{3} - \frac{5}{12} \right)\pi abc = \frac{11}{12}\pi abc.$ ✓

(c) $\forall b>0$ F e' simmetrico rispetto a z.

Essendo l'integranda (z) dispari in z,

si ha $$\int_F z \, dx\,dy\,dz = 0 \qquad \forall b>0.$$

Quindi $$\lim_{b \to +\infty} \int_F z \, dx\,dy\,dz = 0.$$ ■