A VERY SHORT TOUR ON DIFFERENTIAL GAMES

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Contents

These pages are only a schedule of some lessons for the final part of a course on Optimal Control. It is a collection of some things that I use during my lessons: in particular, I would like to mention the very interesting paper due to Bressan [3], the book [2], the sixth chapter of the note by Evans [6] and ....

In this schedule we use the optimal control theory, without recalling the fundamental notions and results: I will use the notations used in [4].

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Chapter 1

Introduction

See [3]

1.1 Concepts of equilibrium in game theory

Game theory deals with situations in which a finite number of players do maximize their own payoff, deciding a strategy among all the available options. Generally each player establishes his own strategy at the same time, taking into account that the game's result depends also on the choice taken by others. Without loss of generality, let's consider the case with two players; both of them have to solve the following problem:

$$\max_{\mathbf{x}_i \in X_i} \quad J_i(\mathbf{x}_1, \mathbf{x}_2) \tag{1.1}$$

where $\mathbf{x}_i \in X_i$, which is the set of all possible options for Player $i = 1, 2$. This is a "one shot game" meaning the payoff is entirely determined by the particular selected strategy.
In general it is not possible to find a solution $(\mathbf{x}_1^*, \mathbf{x}_2^*) \in X_1 \times X_2$ which leads both Player 1 and Player 2 to get the maximum payoff; indeed, the outcome could be favourable only for one of them. This is the reason why there exist different concepts of equilibrium that differ from each other in some features, such as the type of available information, games' mechanism in terms of choices' sequence or, as an alternative, the faculty to cooperate. Hereafter there are the main concepts of solution: Nash equilibrium, Stackelberg equilibrium and Pareto optimality, but first it is necessary to introduce some notations.
In the simplest case of two players, say "Player A" and "Player B" the required ingredients are given by:

- The two (finite/infinite) sets of strategy A and B: the players choose their particular strategy, respectively $a \in A$ and $b \in B$, so that the payoffs achieved are $J_A(a, b)$ and $J_B(a, b)$.
- The two payoff functions: $J^A : A \times B \to \mathbb{R}$ and $J^B : A \times B \to \mathbb{R}$ which are continuous and known by both players.

Definition 1.1 (**Nash equilibrium**). *The pair of strategies (a^*, b^*) is a Nash equilibrium of the game if, for every $a \in A$ and $b \in B$, one has*

$$J_A(a, b^*) \leq J_A(a^*, b^*) \qquad J_B(a^*, b) \leq J_B(a^*, b^*)$$

In this situation none of the players may increase his payoff changing his own strategy if the other do not deviate from his one. This is a solution concept of non-cooperative game.

Definition 1.2 (**Stackelberg equilibrium**). *A pair of strategies $(a^*, b^*) \in A \times B$ is called a Stackelberg equilibrium if $b^* \in R^B(a^*)$ and moreover*

$$J_A(a, b) \leq J_A(a^*, b^*) \qquad \forall (a, b),\ b \in R^B(a),\ a \in A,$$

where $R^B(a)$ is the set of best possible replies of Player B (the follower), since Player A (the leader) has already announced the strategy a, i.e.

$$R^B(a) = \{b' \in B : \; J_B(a,b) \leq J_B(a,b'), \; \forall b \in B\}.$$

Note that b^* stands for the best reply of Player B, which can choose his strategy only after Player A (the leader) has announced his own one. In other words, first Player A establishes his strategy optimizing his utility function, then Player B defines his strategy taking into account what the first player has decided (asymmetry of information).

1.2 Differential games

Let $\mathbf{x} \in \mathbb{R}^n$ describe the state of the system, evolving in time according to the ODE (called dyinamics)

$$\dot{\mathbf{x}}(t) = g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_N), \qquad \text{a.e. } t \in [0,T] \tag{1.2}$$

with fixed $T > 0$ and a initial data

$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n. \tag{1.3}$$

Here $\mathbf{u}_1, \; \mathbf{u}_2, \dots, \mathbf{u}_N$ are the controls of the N players (clearly we suppose $N \geq 2$). We assume that they satisfy the pointwise constraints

$$\mathbf{u}_i(t) \in U_i, \qquad i = 1, \dots N,$$

where $U_i \subset \mathbb{R}^{k_i}$ are the *control sets* for the i-Player.

It is clear that the possibility to solve the Cauchy problem (1.2)–(1.3) is not clear: however, we usually assume that the function g is continuous, differentiable w.r.t. $\mathbf{x}$ and with the derivatives $\frac{\partial g}{\partial x_j}(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_N)$ continuous.

The aim of the i-player is to maximize

$$J_i(\mathbf{u}_1, \dots \mathbf{u}_N) = \int_0^T f_i(t, \mathbf{x}, \mathbf{u}_1, \dots \mathbf{u}_N) \,\mathrm{d}t + \psi_i(T, \mathbf{x}(T)),$$

where, as usual in the control theory, f_i are the running cost and ψ_i are the payoff. Clearly the i-player controls only the choice of $\mathbf{u}_i$.

In all that follows we suppose that there are only two players ($N = 2$), but it is easy to generalize.

The information available to players, such as the current state of the system and the strategy adopted by the competitor, determine the kind of game that has to be undertaken by them. Let's first give some assumptions upon which the following analysis will be established and then let's expose some of the most well-known differential games. Each player has perfect knowledge of:

- the evolution of the system (identified by the function g), and the control sets U_1, U_2.
- the two payoff functions J_1, J_2.
- the instantaneous time $t \in [0,T]$
- the initial condition for the system x_0

1.2.1 Some particular two-persons games

Let us introduce some particular situation for the two-persons games. We say that the game is *symmetric* if

$$f_1(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = f_2(t, \mathbf{x}, \mathbf{u}_2, \mathbf{u}_1),$$
$$\psi_1 = \psi_2, \quad g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = g(t, \mathbf{x}, \mathbf{u}_2, \mathbf{u}_1), \quad U_1 = U_2.$$

A game is *completely cooperative* if

$$f_1 = f_2, \qquad \psi_1 = \psi_2, \qquad U_1 = U_2.$$

A game is *zero-sum* if

$$f_1 = -f_2, \qquad \psi_1 = -\psi_2;$$

in this case, setting $f = f_1$ and $\psi = \psi_1$, the strike of the first player is to find a strategy $\mathbf{u}_1$ in order to

$$\max_{\mathbf{u}_1} J(\mathbf{u}_1, \mathbf{u}_2),$$

where

$$J(\mathbf{u}_1, \mathbf{u}_2) = \int_0^T f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\,\mathrm{d}t + \psi(T, \mathbf{x}(T));$$

while the strike of second player is to minimize the same functional, controlling $\mathbf{u}_2$, since

$$\max_{\mathbf{u}_2} \int_0^T -f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\,\mathrm{d}t - \psi(\mathbf{x}(T)) = -\min_{\mathbf{u}_2} J(\mathbf{u}_1, \mathbf{u}_2).$$

1.2.2 Information structure: open-loop and feedback in two–persons game

In this context are just discussed two differential games: *Open–loop strategies* and *Feedback (or Markovian) strategies*. Essentially, open–loop means that the players base their decision only on time and an initial condition; whereas, the players use the position/state of the game as information basis in a feedback context. A feature that is common to those two information structure is that the players do not need to remember the whole history of the game when making a decision: only running time and the initial position $\mathbf{x}_0$ are relevant for the open–loop information structure, while for the feedback structure, only information on the current position is relevant.

We focus our attention on two–persons game:

Definition 1.3. ***Open–loop strategies***. *The set S_i of strategies available to the i-th Player, with $i = 1, 2$, will consist of all measurable functions $\mathbf{u}_i : [0, T] \to U_i$ such that*

$$\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}_0),$$

where $\mathbf{x}_0$ is the initial data and $\boldsymbol{\nu}_i$ is a decision rule, i.e. a measurable function $\boldsymbol{\nu}_i : [0, T] \times \mathbb{R}^n \to U_i$.

Definition 1.4. ***Feedback strategies*** *(or* ***Markovian strategies****). Here, the control implemented by Player i, for $i = 1, 2$, is $\mathbf{u}_i$, depending on both time t and system's state $\mathbf{x}$. The set S_i of strategies available to the i-th Player will consist of all measurable functions $\mathbf{u}_i : [0, T] \to U_i$ such that*

$$\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}(t)),$$

where $\boldsymbol{\nu}_i$ is a decision rule, i.e. a measurable function $\boldsymbol{\nu}_i : [0, T] \times \mathbb{R}^n \to U_i$; the Player i observes the system's position $(t, \mathbf{x}(t))$ and chooses his action as described decision rule $\boldsymbol{\nu}_i$.

Other concepts of strategies can be given (see for example [2]).

Clearly, as we will see in the next lines, we require that the previous controls are admissible too. To be more precise,

Definition 1.5. ***The class $\mathcal{A}_{OL}$ and the class $\mathcal{A}_{FB}$***. *We say that $(\mathbf{u}_1, \mathbf{u}_2)$, with $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}_0)$, is* an admissible control (or strategy) in the class $\mathcal{A}_{OL}$ of open loop strategies *for the game (2.1) if $t \mapsto (\boldsymbol{\nu}_1(t, \mathbf{x}_0), \boldsymbol{\nu}_2(t, \mathbf{x}_0)) \in U_1 \times U_2$ is a measurable function such that there exists a unique solution $\mathbf{x}$ of the ODE*

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}_0), \boldsymbol{\nu}_2(t, \mathbf{x}_0)) & a.e.\ t \in [0, T] \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

We say that $(\mathbf{u}_1, \mathbf{u}_2)$, *with* $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}(t))$, *is* an admissible control (or strategy) in the class $\mathcal{A}_{FB}$ of feedback strategies *for the game (2.1) if* $(t, \mathbf{x}) \mapsto (\boldsymbol{\nu}_1(t, \mathbf{x}), \boldsymbol{\nu}_2(t, \mathbf{x})) \in U_1 \times U_2$ *is a measurable function such that there exists a unique solution* $\mathbf{x}$ *of the ODE*

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}(t)), \boldsymbol{\nu}_2(t, \mathbf{x}(t))) & a.e.\ t \in [0, T] \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

1.2.3 Target and game set

In the games, as in the problem of optimal control, we deal with trajectories that satisfies some initial and final condition. Let us consider a *target set* $\mathcal{T} \subset \mathbb{R}^+ \times \mathbb{R}^n$; in this note we will consider closed target sets. If we consider a dynamics and a target set, we define the *game set* $\mathcal{G} \subset \mathbb{R}^+ \times \mathbb{R}^n$ as the points $(\tau, \boldsymbol{\xi}) \in [0, \infty) \times \mathbb{R}^n$ such that there exists at least a trajectory $\mathbf{x} : [\tau, T] \times \mathbb{R}^n$ such that

$$\mathbf{x}(\tau) = \boldsymbol{\xi} \qquad \text{and} \qquad (T, \mathbf{x}(T)) \in \partial\mathcal{T};$$

usually we say that $\mathbf{x}$ transfers $(\tau, \boldsymbol{\xi}) \in \mathcal{G}$ in $\mathcal{T}$.

For every $(\tau, \boldsymbol{\xi}) \in \mathcal{T}$, we can consider the trajectory $\mathbf{x} : [\tau, \tau] \to \mathbb{R}^n$ defined by $\mathbf{x}(\tau) = \boldsymbol{\xi}$; this trajectory implies that

$$\mathcal{T} \subset \mathcal{G}.$$

Let us list some particular case of game sets, depending on the final condition on the trajectory:

- (*fixed time and fixed value of the trajectory*): for $\mathbf{x}(T) = \boldsymbol{\beta}$, with T and $\boldsymbol{\beta}$ fixed, we have $\mathcal{T} = \{(T, \boldsymbol{\beta})\}$;
- (*fixed time and free value of the trajectory*): we have $\mathcal{T} = \{T\} \times \mathbb{R}^n$;
- (*free time and fixed value of the trajectory*): we have $\mathcal{T} = \mathbb{R}^+ \times \{\boldsymbol{\beta}\}$, with $\boldsymbol{\beta} \in \mathbb{R}^n$ fixed.

Chapter 2

Nash equilibria for two–persons game

In all this chapter we are considering a two–person game (the general case of a N–person game is similar)

$$\begin{cases} \text{Player I: } \max_{\mathbf{u}_1} J_1(\mathbf{u}_1, \mathbf{u}_2) \qquad \text{Player II: } \max_{\mathbf{u}_2} J_2(\mathbf{u}_1, \mathbf{u}_2) \\ J_i(\mathbf{u}_1, \mathbf{u}_2) = \int_0^T f_i\,(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\,\mathrm{d}t + \psi_i(\mathbf{x}(T)), \quad i = 1, 2 \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{2.1}$$

where T is fixed, U_1 and U_2 are closed control sets for the players and $(\mathbf{u}_1, \mathbf{u}_2)$ is an admissible control, i.e. depending on the information structure.

A Nash equilibrium $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is such that

$$\begin{aligned} J_1(\mathbf{u}_1^*, \mathbf{u}_2^*) &\geq J_1(\mathbf{u}_1, \mathbf{u}_2^*), \qquad \forall \mathbf{u}_1 \\ J_2(\mathbf{u}_1^*, \mathbf{u}_2^*) &\geq J_2(\mathbf{u}_1^*, \mathbf{u}_2), \qquad \forall \mathbf{u}_2, \end{aligned}$$

taking into account the information structure and the admissibility of the controls, as we will study in the next sections.

2.1 Open-loop Nash equilibria

Definition 2.1. *A pair of control functions* $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{OL}$*, with decision rule* $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}_0)$ *and trajectory* $\mathbf{x}^*$ *such that*

$$\begin{cases} \dot{\mathbf{x}}^*(t) = g(t, \mathbf{x}^*(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}_0), \boldsymbol{\nu}_2^*(t, \mathbf{x}_0)) \qquad a.e.\ t \in [0, T] \\ \mathbf{x}^*(0) = \mathbf{x}_0 \end{cases}$$

is a ***Nash equilibrium within the class of open–loop strategies*** $\mathcal{A}_{OL}$ *for the game (2.1) if the following holds:*

I the control $\mathbf{u}_1^*$ *provides a solution to the optimal control problem for the Player I, i.e. for*

$$\begin{cases} \max_{(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{OL}} \int_0^T f_1\,(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}_0), \boldsymbol{\nu}^*(t, \mathbf{x}_0))\,\mathrm{d}t + \psi_1(\mathbf{x}(T)) \\ \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}_0), \boldsymbol{\nu}_2^*(t, \mathbf{x}_0)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

with $\mathbf{u}_1(t) = \boldsymbol{\nu}_1(t, \mathbf{x}(t))$*;*

II the control $\mathbf{u}_2^*$ *provides an optimal open–loop control for the problem for the Player II, i.e. for*

$$\begin{cases} \max_{(\mathbf{u}_1^*, \mathbf{u}_2) \in \mathcal{A}_{OL}} \int_0^T f_2\,(t, \mathbf{x}(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}_0), \boldsymbol{\nu}_2(t, \mathbf{x}_0))\,\mathrm{d}t + \psi_2(\mathbf{x}(T)) \\ \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}_0), \boldsymbol{\nu}_2(t, \mathbf{x}_0)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

with $\mathbf{u}_2(t) = \boldsymbol{\nu}_2(t, \mathbf{x}_0)$.

In order to find a pair of open-loop strategies $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ yielding a Nash equilibrium, it is reasonable to introduce

$$\begin{aligned} H_1(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \boldsymbol{\lambda}_1) &= f_1(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\lambda}_1 \cdot g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \\ H_2(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \boldsymbol{\lambda}_2) &= f_2(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\lambda}_2 \cdot g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \end{aligned} \tag{2.2}$$

A necessary condition for optimality is given by Theorem A.1, taking into account that since T is fixed and the final value of the trajectory is free we can assume that the control is normal (see Theorem 6.13 in [2]):

Theorem 2.1. *Let us consider the problem (2.1) with* f_i, ψ_i *and* g *in* C^1. *Let* $(\mathbf{u}_1^*, \mathbf{u}_2^*)$, *with* $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}_0)$, *be a Nash equilibrium in the class od open–loop strategies. Let* $\mathbf{x}^*$ *be the associated trajectory.*
Then there exists a continuous multiplier $\boldsymbol{\lambda}_i^* : [0, T] \to \mathbb{R}^n$, *with* $i = 1, 2$, *such that*

i) for all $t \in [0, T]$ *we have*

$$\begin{aligned} \mathbf{u}_1^*(t) &\in \arg\max_{\mathbf{v} \in U_1} H_1(t, \mathbf{x}^*(t), \mathbf{v}, \mathbf{u}_2^*(t), \boldsymbol{\lambda}_1^*(t)) \\ \mathbf{u}_2^*(t) &\in \arg\max_{\mathbf{v} \in U_2} H_2(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{v}, \boldsymbol{\lambda}_2^*(t)); \end{aligned}$$

ii) in $[0, T]$ *we have* $\dot{\boldsymbol{\lambda}}_1^* = -\nabla_{\mathbf{x}} H_1(t, \mathbf{x}^*, \mathbf{u}_1^*, \mathbf{u}_2^*, \boldsymbol{\lambda}_1^*)$, $\quad \dot{\boldsymbol{\lambda}}_2^* = -\nabla_{\mathbf{x}} H_2(t, \mathbf{x}^*, \mathbf{u}_1^*, \mathbf{u}_2^*, \boldsymbol{\lambda}_2^*)$;

iii) we have $\boldsymbol{\lambda}_1^*(T) = \nabla_{\mathbf{x}} \psi_1(\mathbf{x}^*(T))$, $\quad \boldsymbol{\lambda}_2^*(T) = \nabla_{\mathbf{x}} \psi_2(\mathbf{x}^*(T))$.

Since Theorem A.1 gives only a necessary condition for optimality, we have to consider sufficient results such as Mangasarian's sufficient conditions (see Theorem A.2) or Arrow's sufficient conditions (see Theorem A.3), in order to be sure the couple of controls $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ stands for an open-loop Nash equilibrium.

2.1.1 Workers versus capitalists

This model is due to Lancaster (see [13]). Let us denote by $k = k(t)$ the capital stock of the economy, and the rate of production is proportional to k, i.e. the production at time t is $\alpha k(t)$, with $\alpha > 0$ fixed. Within the limits a and b, workers decide their share $u = u(t)$ of production; the remaining production $(1-u)\alpha k$ is controlled by the capitalists, who invest a fraction $v = v(t)$ and consume the other portion, i.e. $(1-v)(1-u)\alpha k$. Both workers and capitalists want to maximize their own total consumption.

$$\begin{cases} \text{Workers: } \max\limits_u \displaystyle\int_0^T u\alpha k \, \mathrm{d}t & \text{Capitalists: } \max\limits_v \displaystyle\int_0^T (1-v)(1-u)\alpha k \, \mathrm{d}t \\ 0 < a \le u \le b < 1 & 0 \le v \le 1 \\ \dot{k} = v(1-u)\alpha k & \\ k(0) = k_0 > 0 & \end{cases}$$

Although workers usually do gain future benefits from investments, their willingness to sacrifice consumption can be exploited to the capitalists. On the other hand, a willingness to invest will be less effective if the workers too soon press their share towards the limit b. We choose the time unit such that the constant of proportionality α is 1.

We have the Hamiltonians

$$H_1 = uk + \lambda_1 v(1-u)k, \qquad H_2 = (1-v)(1-u)k + \lambda_2 v(1-u)k,$$

and using Theorem 2.1 we have

$$u \in \arg\max_{p \in [a,b]} kp(1 - \lambda_1 v) = \begin{cases} b & \text{if } \lambda_1 v < 1 \\ ?? & \text{if } \lambda_1 v = 1 \\ a & \text{if } \lambda_1 v > 1 \end{cases} \tag{2.3}$$

$$\begin{aligned}
v &\in \arg\max_{q\in[0,1]} kq(\lambda_2 - 1) = \begin{cases} 1 & \text{if } \lambda_2 > 1 \\ ?? & \text{if } \lambda_2 = 1 \\ 0 & \text{if } \lambda_2 < 1 \end{cases} && (2.4) \\
\dot{k} &= v(1-u)k \\
\dot{\lambda}_1 &= -u - \lambda_1 v(1-u) && (2.5) \\
\dot{\lambda}_2 &= (u-1)(1 - v + \lambda_2 v) && (2.6) \\
\lambda_1(T) &= 0 && (2.7) \\
\lambda_2(T) &= 0 && (2.8)
\end{aligned}$$

where in order to obtain (2.3) and (2.4) we use that $k(t) \geq k_0 > 0$ since $\dot{k} \geq 0$. We note that u and v in (2.3) and (2.4) do not depend on the trajectory k: hence we are in the position to looking for a open–loop solution.
It is easy to see that (2.4) implies that in (2.6) we have

$$\dot{\lambda}_2(t) < 0, \qquad \forall t \in [0,T]: \tag{2.9}$$

indeed by the adjoint equation (2.6) and the Maximum principle (2.4),

$$\begin{aligned}
&\text{if } \lambda_2 > 1, && \Rightarrow \dot{\lambda}_2 = (u-1)\lambda_2 < 0 \\
&\text{if } \lambda_2 \leq 1, && \Rightarrow \dot{\lambda}_2 = (u-1) < 0.
\end{aligned}$$

Hence, by (2.8), there exists $\tau \in [0,T)$ such that

$$\lambda_2(t) < 1 \qquad \forall t \in (\tau, T]. \tag{2.10}$$

This implies, by (2.3) and (2.4), $v(t) = 0$ and $u(t) = b$ in $(\tau, T]$. Relations (2.5)–(2.8) give

$$\lambda_1(t) = -b(t-T), \qquad \lambda_2(t) = (1-b)(T-t) \quad \forall t \in (\tau, T]; \tag{2.11}$$

Condition (2.10) implies

$$\tau = T - \frac{1}{1-b}. \tag{2.12}$$

Note that $\lambda_2(\tau) = 1$ and together with (2.9) we have $\lambda_2(t) > 1$ in $[0,\tau)$: hence, by (2.4), we obtain

$$v(t) = 1 \qquad \forall t \in [0, \tau]. \tag{2.13}$$

Now we have to distinguish two cases: $b \geq \frac{1}{2}$ and $b < \frac{1}{2}$.

• $b \geq \frac{1}{2}$: Note that for such b, by (2.11), we have $\lambda_1(\tau) = \frac{b}{1-b} \geq 1$; moreover, by in (2.5) we obtain $\dot{\lambda}_1(\tau) < 0$. This gives that $\lambda_1(t) > 1$ for $t \in [\tau - \varepsilon, \tau)$ for some positive ε: now, replacing the same arguments we obtain that $\lambda_1(t) > 1$ in $[0, \tau]$. Hence, this inequality and (2.13) give by (2.3) that $u(t) = a$ in $[0,\tau]$. We obtain that the candidate to be a Nash equilibrium is (u^*, v^*) with

$$u^*(t) = \begin{cases} a & \text{if } t \in [0,\tau] \\ b & \text{if } t \in (\tau, T] \end{cases}, \qquad v^*(t) = \begin{cases} 1 & \text{if } t \in [0,\tau] \\ 0 & \text{if } t \in (\tau, T] \end{cases}$$

with τ as in (2.12). First, we have to guarantee that (u^*, v^*) is admissible, i.e. there exists a unique path k^*, solution of the dynamics and the initial condition. We have that

$$\begin{cases} \dot{k} = v^*(1-u^*)k = (1-a)k & \text{if } t \in [0,\tau] \\ k(0) = k_0 \end{cases}$$

gives $k(t) = k_0 e^{(1-b)t}$, for $t \in [0, \tau]$; moreover

$$\begin{cases} \dot{k} = v^*(1-u^*)k = 0 & \text{if } t \in [\tau, T] \\ k(\tau) = k_0 e^{(1-b)\tau} \end{cases}$$

gives $k(t) = k_0 e^{(1-b)\tau}$, for $t \in [\tau, T]$. Hence (u^*, v^*) is admissible.
In order to prove that (u^*, v^*) is really a Nash equilibrium, we remark that $(k,u) \mapsto H_1(t,k,u,v^*,\lambda_1^*)$ and $(k,v) \mapsto H_2(t,k,u^*,v,\lambda_2^*)$ are not concave functions (for fixed t) and hence we are not in the position to apply Theorem A.2; however, if we construct the maximized Hamiltonian functions H_1^0 and H_2^0 for H_1 and H_2 respectively, we obtain

$$H_1^0(t,k,v^*,\lambda_1^*) = \max_{u \in [a,b]} H_1(t,k,u,v^*,\lambda_1^*) = k[\lambda_1^* v^* + \max_{u\in[a,b]} u(1-\lambda_1^*)v^*)],$$
$$H_2^0(t,k,u^*,\lambda_2^*) = \max_{v \in [0,1]} H_2(t,k,u^*,v,\lambda_2^*) = k(1-u^*)[1 + \max_{v\in[0,1]} (\lambda_2^* - 1)v)];$$

it is easy to verify that such two functions are concave in k, for fixed t, and hence Theorem A.3 guarantees that (u^*, v^*) is a Nash equilibrium.

- $b < \frac{1}{2}$: Note that $\lambda_1(\tau) = \frac{b}{1-b} < 1$; then there exists $\tau' \in [0, \tau)$ such that

$$\lambda_1(t) < 1 \qquad \forall t \in [\tau', \tau]. \tag{2.14}$$

This inequality with (2.13) imply that, by (2.3), $u(t) = b$. The adjoint equation (2.5) and the condition for λ_1 in τ give the ODE

$$\begin{cases} \dot{\lambda}_1 = -b - \lambda_1(1-b) & \text{for } t \in [\tau', \tau], \\ \lambda_1(\tau) = \frac{b}{1-b} \end{cases}$$

The solution is

$$\lambda_1(t) = \frac{2b}{1-b} e^{-(1-b)(t-\tau)} - \frac{b}{1-b}.$$

It is easy to see that for

$$\tau' = \tau + \frac{1}{1-b} \ln(2b) \tag{2.15}$$

we have $\lambda_1(\tau') = 1$. Now the same argument of the case $b \geq 1/2$ gives that $\lambda_1(t) > 1$ in $[0, \tau']$ and we obtain, as before, $u(t) = a$. Hence we have that the candidate to be a Nash equilibrium is (u^*, v^*),

$$u^*(t) = \begin{cases} a & \text{if } t \in [0, \tau'] \\ b & \text{if } t \in (\tau', T] \end{cases}, \qquad v^*(t) = \begin{cases} 1 & \text{if } t \in [0, \tau] \\ 0 & \text{if } t \in (\tau, T] \end{cases}$$

with τ and τ' as in (2.12) and (2.15). In order to prove that (u^*, v^*) is really a Nash equilibrium, we use arguments similar to the previous case.

2.2 Feedback Nash equilibria

Definition 2.2. *A pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, with decision rule $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$ and trajectory $\mathbf{x}^*$ such that*

$$\begin{cases} \dot{\mathbf{x}}^*(t) = g(t, \mathbf{x}^*(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}^*(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}^*(t))) & \text{a.e. } t \in [0,T] \\ \mathbf{x}^*(0) = \mathbf{x}_0 \end{cases}$$

*is a **Nash equilibrium within the class of feedback strategies** $\mathcal{A}_{FB}$ for the game (2.1) if the following holds:*

I the control $\mathbf{u}_1^*$ provides an optimal feedback control to the problem for the first Player, i.e. for

$$\begin{cases} \max\limits_{(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{FB}} \int_0^T f_1(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}(t)))\, \mathrm{d}t + \psi_1(\mathbf{x}(T)) \\ \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}(t))) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

where $\mathbf{u}_1(t) = \boldsymbol{\nu}_1(t, \mathbf{x}(t))$;

II the control $\mathbf{u}_2^*$ *provides an optimal feedback control to the problem for the second Player, i.e. for*

$$\begin{cases} \max_{(\mathbf{u}_1^*,\mathbf{u}_2)\in\mathcal{A}_{FB}} \int_0^T f_2\left(t,\mathbf{x}(t),\boldsymbol{\nu}_1^*(t,\mathbf{x}(t)),\boldsymbol{\nu}_2(t,\mathbf{x}(t))\right)\,\mathrm{d}t + \psi_2(\mathbf{x}(T)) \\ \dot{\mathbf{x}}(t) = g(t,\mathbf{x}(t),\boldsymbol{\nu}_1^*(t,\mathbf{x}(t)),\boldsymbol{\nu}_2(t,\mathbf{x}(t))) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

where $\mathbf{u}_2(t) = \boldsymbol{\nu}_2(t,\mathbf{x}(t))$.

Variational approach is not useful

Let us suppose that we are interesting on finding a feedback solution using the variational method: we will show that the Pontryagin necessary condition is much more complicated and it is not useful. In order to do that, let us suppose that $(\mathbf{u}_1^*,\mathbf{u}_2^*)\in\mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t)=\boldsymbol{\nu}_i^*(t,\mathbf{x}^*(t))$, is a Nash equilibrium within the class of feedback strategies, with $\mathbf{x}^*$ its trajectory. Without lost of generality, we assume that $n=k_1=k_2=1$; moreover, we assume $U_1=U_2=\mathbb{R}$. Hence we have $(t,x)\mapsto(\nu_1^*(t,x),\nu_2^*(t,x))\in\mathbb{R}^2$ measurable function such that x^* is the solution of the ODE

$$\begin{cases} \dot{x}(t) = g(t,x(t),\nu_1^*(t,x(t)),\nu_2^*(t,x(t))) \\ x(0)=x_0 \end{cases}$$

and $u_1^*(t)=\nu_1^*(t,x^*(t)),\ u_2^*(t)=\nu_2^*(t,x^*(t))$.

Let us put our attention on the first Player and we fix a continuous function $h:[0,T]\to\mathbb{R}$ and for every constant $\epsilon\in\mathbb{R}$ we define the function $u_{1,\epsilon}:[0,T]\to\mathbb{R}$ by

$$u_{1,\epsilon}(t)=\nu_1^*(t,x^*(t))+\epsilon h(t)=u_1^*(t)+\epsilon h(t) \tag{2.16}$$

and we suppose that there exists the trajectory x_ϵ associated to $(u_{1,\epsilon},\nu_2^*)$, i.e. the solution of the ODE

$$\begin{cases} \dot{x}(t) = g(t,x(t),u_{1,\epsilon}(t),\nu_2^*(t,x(t))) \\ x(0)=x_0 \end{cases}$$

Clearly

$$u_1^*(t)=u_{1,0}(t) \qquad x_0(t)=x^*(t), \qquad x_\epsilon(0)=x_0. \tag{2.17}$$

As usual in the variational approach to a problem of optimal control, we define the function $\mathcal{J}_h:\mathbb{R}\to\mathbb{R}^2$ as

$$\mathcal{J}_h(\epsilon)=\int_0^T f_1(t,x_\epsilon(t),u_{1,\epsilon}(t),\nu_2^*(t,x_\epsilon(t)))\,\mathrm{d}t+\psi_1(x_\epsilon(T))$$

We introduce the Hamiltonian H_1 as in (2.2). Using the dynamics and by integrating by part we have

$$\mathcal{J}_h(\epsilon)=\int_0^T\left[H_1(t,x_\epsilon,u_{1,\epsilon},\nu_2^*(t,x_\epsilon),\lambda_1)+\dot{\lambda}_1x_\epsilon\right]\mathrm{d}t-\left(\lambda_1x_\epsilon\Big|_0^T+\psi_1(x_\epsilon(T))\right)$$

Since (u_1^*,u_2^*) is a Nash equilibrium, for the first Player we have that optimal $\mathcal{J}_h(0)\ge\mathcal{J}_h(\epsilon)$, for every ϵ, and hence $\frac{\mathrm{d}\mathcal{J}_h}{\mathrm{d}\epsilon}(0)=0$. Classical calculation gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}\mathcal{J}_h}{\mathrm{d}\epsilon}(0) \\ &= \int_0^T\left\{\left[\frac{\partial H_1}{\partial x}(t,x^*,u_1^*,u_2^*,\lambda_1)+\dot{\lambda}_1+\frac{\partial H_1}{\partial u_2}(t,x^*,u_1^*,u_2^*,\lambda_1)\frac{\partial\nu_2^*}{\partial x}(t,x^*)\right]\frac{\mathrm{d}x_\epsilon}{\mathrm{d}\epsilon}(0)+\right. \\ &\qquad \left.+\frac{\partial H_1}{\partial u_1}(t,x^*,u_1^*,u_2^*,\lambda_1)h\right\}\mathrm{d}t-\left[\lambda_1(T)-\frac{\partial\psi_1}{\partial x}(x^*(T))\right]\frac{\mathrm{d}x_\epsilon(T)}{\mathrm{d}\epsilon}(0) \end{aligned}$$

We note that the bad new is the term $\frac{\partial H_1}{\partial u_2}(t,x^*,u_1^*,u_2^*,\lambda_1)\frac{\partial\nu_2^*}{\partial x}(t,x^*)$ that arrives from the fact that we are working with feedback controls, i.e. $\nu_2^*(t,x(t))$. In [9] (see Theorem 7.1) there is a sufficient condition for a particular type of games in order to obtain a feedback Nash equilibrium using the variational approach: such condition it not really useful.

With the Dynamic Programming approach

Let us start with the definition of the value function that, with respect to the situation of optimal control problems, it must be specialized:

Definition 2.3. *Let's suppose that a pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i(t, \mathbf{x}^*(t))$, is a Nash equilibrium within the class of feedback strategies $\mathcal{A}_{FB}$ for the game (2.1). Then we define the value functions $V_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, for Player i, by*

$$V_1(\tau, \boldsymbol{\xi}) = \sup_{(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{FB}} \int_\tau^T f_1(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}(t)))\, \mathrm{d}t + \psi_1(\mathbf{x}(T))$$

where $\mathbf{u}_1(t) = \boldsymbol{\nu}_1(t, \mathbf{x}(t))$. Similarly, we define

$$V_2(\tau, \boldsymbol{\xi}) = \sup_{(\mathbf{u}_1^*, \mathbf{u}_2) \in \mathcal{A}_{FB}} \int_\tau^T f_2(t, \mathbf{x}(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}(t)), \boldsymbol{\nu}_2(t, \mathbf{x}(t)))\, \mathrm{d}t + \psi_2(\mathbf{x}(T))$$

where $\mathbf{u}_2(t) = \boldsymbol{\nu}_2(t, \mathbf{x}(t))$.

Some comments: first we remark that such definition is on a Nash equilibrium. Second, to be clear, in the definition of V_1 we consider the sup on all the feedback controls $\mathbf{u}_1$ such that, for the fixed feedback control $\mathbf{u}_2^*$, there exists a unique solution of

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1(t, \mathbf{x}(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}(t))) & \text{a.e. in } [\tau, T] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

Finally, since $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a Nash equilibrium, we have that

$$V_1(\tau, \boldsymbol{\xi}) = \int_\tau^T f_1(t, \mathbf{x}, \mathbf{u}_1^*, \mathbf{u}_2^*)\, \mathrm{d}t + \psi_1(\mathbf{x}(T))$$

Similar situation appears in definition of V_2.

Now, we are in the position to apply the Dynamic Programming results. Let us suppose that the value functions V_i, defined on a feedback Nash equilibrium $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ with trajectory $\mathbf{x}^*$, are continuosly differentiable. The idea is to write the Bellman-Hamilton-Jacobi system for the first player with its value function V_1 and the Bellman-Hamilton-Jacobi system for the second player with its value function V_2, i.e.

$$\begin{cases} \dfrac{\partial V_1}{\partial t}(t, \mathbf{x}) + \max\limits_{\mathbf{v}_1 \in U_1} \Big[f_1(t, \mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t, \mathbf{x})) + \nabla_{\mathbf{x}} V_1(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t, \mathbf{x})) \Big] = 0, & \forall (t, \mathbf{x}) \in [0,T] \times \mathbb{R}^n \\ \dfrac{\partial V_2}{\partial t}(t, \mathbf{x}) + \max\limits_{\mathbf{v}_2 \in U_2} \Big[f_2(t, \mathbf{x}, \boldsymbol{\nu}_1^*(t, \mathbf{x}), \mathbf{v}_2) + \nabla_{\mathbf{x}} V_2(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \boldsymbol{\nu}_1^*(t, \mathbf{x}), \mathbf{v}_2) \Big] = 0, & \forall (t, \mathbf{x}) \in [0,T] \times \mathbb{R}^n \\ V_1(T, \mathbf{x}) = \psi_1(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n \\ V_2(T, \mathbf{x}) = \psi_2(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n \end{cases}$$

A version of Theorem A.6 in this context is the following (see Theorem 6.16 in [2])

Theorem 2.2. *Let us consider the us consider the problem (2.1) with f_i, ψ_i and g in C^1. Let's consider a pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$ and $\mathbf{x}^*$ is the solution of the ODE*

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \boldsymbol{\nu}_1^*(t, \mathbf{x}(t)), \boldsymbol{\nu}_2^*(t, \mathbf{x}(t))) & \textit{a.e. in } [0, T] \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Let us suppose that there exist two functions $W_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, continuously differentiable such that, for every $(t, \mathbf{x}) \in [0,T] \times \mathbb{R}^n$,

$$-\frac{\partial W_1}{\partial t}(t, \mathbf{x}) \quad = \quad \max_{\mathbf{v}_1 \in U_1} \Big[f_1(t, \mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t, \mathbf{x})) + \nabla_{\mathbf{x}} W_1(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t, \mathbf{x})) \Big]$$

$$
\begin{aligned}
&= f_1(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} W_1(t,\mathbf{x}) \cdot g(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) \\
-\frac{\partial W_2}{\partial t}(t,\mathbf{x}) &= \max_{\mathbf{v}_2 \in U_2} \Big[f_2(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\mathbf{v}_2) + \nabla_{\mathbf{x}} W_2(t,\mathbf{x}) \cdot g(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\mathbf{v}_2) \Big] \\
&= f_2(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} W_2(t,\mathbf{x}) \cdot g(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) \\
W_1(T,\mathbf{x}) &= \psi_1(\mathbf{x}) \\
W_2(T,\mathbf{x}) &= \psi_2(\mathbf{x})
\end{aligned}
$$

Then $\mathbf{x}^*$ is the optimal trajectory and $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a Nash equilibrium in the class $\mathcal{A}_{FB}$.

2.2.1 Affine–Quadratic differential games

Now let us consider a particular type of games: we say that a two person differential games is Linear–Quadratic if

$$
\begin{cases}
\max\limits_{\mathbf{u}_1} \dfrac{1}{2}\displaystyle\int_0^T \left(\mathbf{x}'Q_1\mathbf{x} + 2\mathbf{x}'S_1 + \mathbf{u}_1'R_{1,1}\mathbf{u}_1 + \mathbf{u}_2'R_{1,2}\mathbf{u}_2\right)\, \mathrm{d}t + \dfrac{1}{2}\mathbf{x}(t_1)'P_1\mathbf{x}(t_1) \\
\max\limits_{\mathbf{u}_2} \dfrac{1}{2}\displaystyle\int_0^T \left(\mathbf{x}'Q_2\mathbf{x} + 2\mathbf{x}'S_2 + \mathbf{u}_1'R_{2,1}\mathbf{u}_1 + \mathbf{u}_2'R_{2,2}\mathbf{u}_2\right)\, \mathrm{d}t + \dfrac{1}{2}\mathbf{x}(t_1)'P_2\mathbf{x}(t_1) \\
g(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) = A\mathbf{x} + B_1\mathbf{u}_1 + B_2\mathbf{u}_2 + C \\
\mathbf{x}(0) = \boldsymbol{\alpha} \\
\mathcal{C} = \{(\mathbf{u}_1,\mathbf{u}_2) : [0,T] \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2},\ \text{admissible}\}
\end{cases}
$$

where $\mathbf{v}'$ is the transpose of the matrix $\mathbf{v}$; we denote the trajectory $\mathbf{x}$ and the control $\mathbf{u}$ such that $\mathbf{x} = (x_1, x_2, \dots, x_n)'$ and $\mathbf{u}_i = (u_{i,1}, u_{i,2}, \dots, u_{i,k_i})'$ respectively; with $Q_i = Q_i(t)$ and $P_i = P_i(t)$ symmetric matrices, and $R_{i,j} = R_{i,j}(t)$, $A = A(t)$, $B_i = B_i(t)$ and $C = C(t)$ matrices. We have the following result (see [2]):

Proposition 2.1. *Let us suppose that for a Linear–Quadratic two person differential games there exist the value functions V_i, then we have*

$$
V_i(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}'Z_i\mathbf{x} + \mathbf{x}W_i + Y_i \tag{2.18}
$$

for $i = 1,2$, with $Z_i = Z_i(t)$, $W_i = W_i(t)$ and $Y_i = Y_i(t)$ matrices.

Moreover, let us mention the following particular situation (see [2], [3] for details):

Remark 2.1. *Let us consider the Linear–Quadratic two person differential games in the linear and homogeneous case, i.e. with*

$$
C = 0, \qquad \text{and} \qquad S_i = 0.
$$

If there exists the value functions for the problem, then

$$
V_i(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}'Z_i(t)\mathbf{x}.
$$

2.2.2 Infinite horizon case

Let us consider a two–person, infinite horizon with discount, differential game

$$
\begin{cases}
\text{Player I: } \max\limits_{\mathbf{u}_1} J_1(\mathbf{u}_1,\mathbf{u}_2) \qquad \text{Player II: } \max\limits_{\mathbf{u}_2} J_2(\mathbf{u}_1,\mathbf{u}_2) \\
J_i(\mathbf{u}_1,\mathbf{u}_2) = \displaystyle\int_0^\infty f_i\left(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2\right) e^{-rt} dt, \quad i = 1,2 \\
\dot{\mathbf{x}} = g(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) \\
\mathbf{x}(0) = \mathbf{x}_0
\end{cases} \tag{2.19}
$$

where $r \geq 0$ and U_i are the two control sets for the players. In this situation we are in the position to introduce the current value functions $V_i^c : \mathbb{R}^n \to \mathbb{R}^n$. More precisely, since f and g do not depend on t, it is possible to prove (as in a optimal control problem) that the existence of the value function (V_1, V_2) that satisfies the BHJ system is equivalent to the existence of the current value function (V_1^c, V_2^c) that satisfies a current BHJ system: moreover

$$V_i(t,\mathbf{x}) = e^{-rt} V_i^c(\mathbf{x}), \qquad \forall (t,\mathbf{x}) \in [0,\infty) \times \mathbb{R}^n$$

Taking into account that

$$\nabla_{\mathbf{x}} V_i(t,\mathbf{x}) = e^{-rt} \nabla_{\mathbf{x}} V_i^c(\mathbf{x}), \qquad \frac{\partial V_i}{\partial t}(t,\mathbf{x}) = -re^{-rt} V_i^c(\mathbf{x}),$$

Theorem 2.2 becomes

Remark 2.2. *Let us consider the us consider the problem (2.19) with f_i and and g in C^1. Let's consider a pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$ and $\mathbf{x}^*$ is the unique solution of the ODE*

$$\begin{cases} \dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \boldsymbol{\nu}_1^*(t,\mathbf{x}(t)), \boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) & a.e.\ in\ [0,\infty) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Let us suppose that there exists two functions $V_i^c : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, continuously differentiable such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} rV_1^c(\mathbf{x}) &= \max_{\mathbf{v}_1 \in U_1} \Big[f_1(\mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} V_1^c(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{v}_1, \boldsymbol{\nu}_2^*(t,\mathbf{x})) \Big] \\ &= f_1(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} V_1^c(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \boldsymbol{\nu}_2^*(t,\mathbf{x})) \\ rV_2^c(\mathbf{x}) &= \max_{\mathbf{v}_2 \in U_2} \Big[f_2(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \mathbf{v}_2) + \nabla_{\mathbf{x}} V_2^c(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \mathbf{v}_2) \Big] \\ &= f_2(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} V_2^c(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(t,\mathbf{x}), \boldsymbol{\nu}_2^*(t,\mathbf{x})) \end{aligned}$$

Then $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a Nash equilibrium in the class $\mathcal{A}_{FB}$.

In many situation of the previous remark we have that the decision rule does not depend explicity by t, i.e.

$$\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(\mathbf{x}(t)). \tag{2.20}$$

We mention that in the case of problem (2.19) one can decide to restrict the attention only to the feedback control of the type (2.20), called stationary feedback strategies (see [14]).

2.2.3 Two firms in competition

Suppose two firms produce an identical product. The cost of producing is governed by the total cost function

$$C(u_i) = cu_i + \frac{1}{2} u_i^2,$$

where $u_i = u_i(t)$ refers to the i-firm's production level at time t and c is a positive constant. Each firm sells all it produces at time t into a market with a common price $p = p(t)$. The relationship between the total amount of production $u_1 + u_2$ supplied and the change in price is described by

$$\dot{p} = s(a - u_1 - u_2 - p),$$

where s and a are positive constants and p_0 is the price at the initial time $t = 0$. Hence the situation is (this model is presented in see [11], page 278)

$$\begin{cases} \text{I Prod.:} \ \max_{u_1} \displaystyle\int_0^\infty e^{-rt} \left(pu_1 - cu_1 - \frac{1}{2} u_1^2 \right) \mathrm{d}t \qquad u_1 \geq 0 \\ \qquad \text{II Prod.:} \ \max_{u_2} \displaystyle\int_0^\infty e^{-rt} \left(pu_2 - cu_2 - \frac{1}{2} u_2^2 \right) \mathrm{d}t \qquad u_2 \geq 0 \\ \qquad \dot{p} = s(a - u_1 - u_2 - p) \\ \qquad p(0) = p_0 > 0 \qquad p(t) \geq 0 \end{cases}$$

with the rate of discount r that is a positive constant. Note that it is a symmetric game. We are interested on a non zero Nash equilibrium in the family of feedback strategies, i.e. strategies for the two firms that depend, at every time, on the price $p(t)$.

With a infinite and discounted problems, it is convenient to introduce the current value functions $V_1^c = V_1^c(p)$ and $V_2^c = V_2^c(p)$ and their Bellman–Hamilton–Jacobi equations: for every p we have

$$
\begin{aligned}
&-rV_1^c + \max_{v\geq 0}\left[pv - cv - \frac{1}{2}v^2 + s(V_1^c)'(a - v - \nu_2 - p)\right] = 0\\
&\qquad \Rightarrow\ -rV_1^c + s(a - \nu_2 - p)(V_1^c)' + \max_{v\geq 0}\left[(p - c - s(V_1^c)')v - \frac{1}{2}v^2\right] = 0\\
&-rV_2^c + \max_{v\geq 0}\left[pv - cv - \frac{1}{2}v^2 + s(V_2^c)'(a - \nu_1 - v - p)\right] = 0\\
&\qquad \Rightarrow\ -rV_2^c + s(a - \nu_1 - p)(V_2^c)' + \max_{v\geq 0}\left[(p - c - s(V_2^c)')v - \frac{1}{2}v^2\right] = 0
\end{aligned}
$$

Clearly we obtain, for $i = 1, 2$

$$
\nu_i^*(t,p) = \nu_i^*(p) = \begin{cases} 0 & \text{if } p - c - s(V_i^c)'(p) \leq 0 \\ p - c - s(V_i^c)'(p) & \text{if } p - c - s(V_i^c)'(p) > 0 \end{cases}
$$

Note that such strategies are stationary feedback strategies: hence if our strategy is zero, for some t, we have that such strategy is zero at every time. Let's concentrate our attention on strategies that are different from zero: hence we have

$$
\nu_1^*(p) = p - c - s(V_1^c)'(p), \qquad \nu_2^*(p) = p - c - s(V_2^c)'(p). \tag{2.21}
$$

in the assumption

$$
p - c - s(V_i^c)'(p) > 0, \qquad i = 1, 2 \tag{2.22}
$$

Let us consider the first current BHJ equation; we obtain

$$
-rV_1^c + s\Big(a - 2p + c + s(V_2^c)'\Big)(V_1^c)' + \frac{1}{2}\Big(p - c - s(V_1^c)'\Big)^2 = 0, \qquad \forall p.
$$

Since the problem is Linear-Quadratic (see Proposition 2.1), we looking for value functions as

$$
V_1^c(p) = \alpha_1 + \beta_1 p + \frac{1}{2}\gamma_1 p^2, \qquad V_2^c(p) = \alpha_2 + \beta_2 p + \frac{1}{2}\gamma_2 p^2. \tag{2.23}
$$

We obtain, using (2.21) and (2.23), that the two current BHJ equations now require that for every p

$$
-r\left(\alpha_1 + \beta_1 p + \frac{1}{2}\gamma_1 p^2\right) + s\Big[a - 2p + c + s\left(\beta_2 + \gamma_2 p\right)\Big]\left(\beta_1 + \gamma_1 p\right) + \frac{1}{2}\Big(p - c - s\left(\beta_1 + \gamma_1 p\right)\Big)^2 = 0 \tag{2.24}
$$

$$
-r\left(\alpha_2 + \beta_2 p + \frac{1}{2}\gamma_2 p^2\right) + s\Big[a - 2p + c + s\left(\beta_1 + \gamma_1 p\right)\Big]\left(\beta_2 + \gamma_2 p\right) + \frac{1}{2}\Big(p - c - s\left(\beta_2 + \gamma_2 p\right)\Big)^2 = 0 \tag{2.25}
$$

The previous equations give two polynomials of degree 2 in p and they are identically zero for every p: equating the coefficients of p^2, we obtain

$$
\begin{aligned}
s^2\gamma_1^2 + (-r - 6s + 2s^2\gamma_2)\gamma_1 + 1 = 0\\
s^2\gamma_2^2 + (-r - 6s + 2s^2\gamma_1)\gamma_2 + 1 = 0
\end{aligned} \tag{2.26}
$$

Let us prove that $\gamma_1 = \gamma_2$: in order to do that, let us subtract the previous two equation obtaining

$$
(\gamma_1 - \gamma_2)[s^2(\gamma_1 + \gamma_2) - r - 6s] = 0.
$$

If $\gamma_1 \neq \gamma_2$, we have

$$s^2(\gamma_1 + \gamma_2) = r + 6s \tag{2.27}$$

Now let us consider the dynamic: with $u_1 = \nu_1^*$ and $u_2 = \nu_2^*$ given in (2.21) and taking into account (2.23), we obtain

$$\dot{p} = s[s(\gamma_1 + \gamma_2) - 3]p + s[a + 2c + s(\beta_1 + \beta_2)].$$

The solution of this ODE in p is

$$p(t) = Ae^{s[s(\gamma_1+\gamma_2)-3]t} - \frac{a + 2c + s(\beta_1 + \beta_2)}{s(\gamma_1 + \gamma_2) - 3} \tag{2.28}$$

where A is a constant that depends on p_0. We note that, by (2.27),

$$s(\gamma_1 + \gamma_2) - 3 = \frac{r + 3s}{s} > 0.$$

Hence, $A \neq 0$, for the price $p(t)$ goes to $\mathrm{sgn}(A) \cdot \infty$, for $t \to \infty$: this is not reasonable. If $A = 0$, then $p(t) = p_0$ is constant; relation (2.21) give $u_i^*(t) = \nu_i^*(p_0)$ and this is not a feedback strategy. Hence $\gamma_1 \neq \gamma_2$ is impossible.

From now on, let us simplify the notations setting $\gamma = \gamma_1 = \gamma_2$. The price, by (2.28), now is

$$p(t) = Ae^{s(2s\gamma-3)t} - \frac{a + 2c + s(\beta_1 + \beta_2)}{2s\gamma - 3} \tag{2.29}$$

and equation (2.27) becomes

$$3s^2\gamma^2 - (r + 6s)\gamma + 1 = 0$$

with solutions

$$\gamma_\pm = \frac{r + 6s \pm \sqrt{(r + 6s)^2 - 12s^2}}{6s^2}.$$

We note that

$$\gamma_+ > \frac{6s + \sqrt{36s^2 - 12s^2}}{6s^2} = \frac{3 + \sqrt{6}}{3s} > \frac{3}{2s}$$

implies again that the price in (2.29) goes to ∞ for $t \to \infty$. Hence we consider only the solution $\gamma = \gamma_-$ and we obtain

$$\gamma_- = \frac{r + 6s - \sqrt{(r + 6s)^2 - 12s^2}}{6s^2} \tag{2.30}$$

Let us prove that $\beta_1 = \beta_2$. Let us note that

$$0 < s^2\gamma_- < 6s^2\gamma_- < r + (6 - 2\sqrt{6})s < r + 3s. \tag{2.31}$$

Equating the coefficients of p in (2.24) and (2.25) we obtain

$$\begin{aligned} (2s^2\gamma_- - r - 3s)\beta_1 + s^2\gamma_-\beta_2 &= c - 2sc\gamma_- - sa\gamma_- \\ s^2\gamma_-\beta_1 + (2s^2\gamma_- - r - 3s)\beta_2 &= c - 2sc\gamma_- - sa\gamma_- \end{aligned} \tag{2.32}$$

Let us subtract the previous two equation obtaining

$$(\beta_1 - \beta_2)(s^2\gamma_- - r - 3s) = 0.$$

This relation, by (2.31), gives $\beta_1 = \beta_2$. Let us set $\beta = \beta_1 = \beta_2$: by (2.32)

$$\beta = \frac{s\gamma_-(2c + a) - c}{r + 3s - 3s^2\gamma_-}. \tag{2.33}$$

It is easy to see that $\beta > 0$. Moreover, by (2.21) and (2.23) we have

$$\nu_1^*(p) = \nu_2^*(p) = p - c - s(\beta + \gamma_- p). \tag{2.34}$$

Now let us prove that $\alpha_1 = \alpha_2$ (note that such coefficients play no role in the strategies, but we have to guarantee that exist (V_1^c, V_2^c) solutions of the current BHJ equations). Taking into account that $\beta_1 = \beta_2 = \beta$, equating the coefficients of zero degree on p in (2.24) and (2.25) we obtain

$$-r\alpha_i + s(a + c + s\beta)\beta + \frac{1}{2}(c + s\beta)^2 = 0, \qquad i = 1, 2.$$

Clearly such α_i exist and are equal. Finally, we obtain by (2.34)

$$u_i^*(t) = \nu_i^*(p^*(t)) = (1 - s\gamma_-)p^*(t) - (\beta s + c), \qquad i = 1, 2 \tag{2.35}$$

$$p^*(t) = \left(p_0 - \frac{a + 2c + 2s\beta}{3 - 2s\gamma_-}\right) e^{s(2s\gamma_- - 3)t} + \frac{a + 2c + 2s\beta}{3 - 2s\gamma_-} \tag{2.36}$$

where γ_- and β are defined in (2.30) and (2.33) respectively.

Let us set $\widetilde{k} = \frac{a+2c+2s\beta}{3-2s\gamma_-}$. It is easy to see, using $\beta > 0$, that $\widetilde{k} > 0$ and hence the shape of the trajectory–price function in (2.36) implies that

$$p^*(t) \geq \min\left(p^*(0), \lim_{t\to\infty} p^*(t)\right) = \min\left(p_0, \widetilde{k}\right) > 0.$$

Hence p^* is a good price, i.e. $p^*(t) \geq 0$. Finally, the assumption (2.22) is now

$$(1 - s\gamma_-)p > (\beta s + c) :$$

this condition requires that the trajectory p lies in a region[1] R

$$R = \{(t, p) \in [0, \infty) \times (0, \infty) : \ (1 - s\gamma_-)p > (\beta s + c)\}.$$

Clearly R depends on the constants involved in the model. More precisely, our trajectory p^* lies is this region R if and only if, again by the shape of p^* in (2.36), the following

$$\min\left(p^*(0), \lim_{t\to\infty} p^*(t)\right) = \min\left(p_0, \widetilde{k}\right) > \frac{\beta s + c}{1 - s\gamma_-} \tag{2.37}$$

is satisfied. Some computations gives that for some choice of (a, c, r, s, p_0) the previous condition holds and for some others choice is not true. However, if (2.37) is satisfied, then Remark 2.2 guarantees that (u_1^*, u_2^*) in (2.35) is a Nash equilibrium in the family of feedback strategies.

2.3 Further examples and models

2.3.1 Two fishermen at the lake

In the present model (see [11], page 285) we will see that the open–loop Nash equilibrium and the feedback Nash equilibrium coincide.

The model. Suppose that the evolution of the stock of fish $x = x(t)$ in a lake is governed by

$$\dot{x} = \alpha x - \beta x \ln x$$

for $t \geq 0$ and where α and β are positive constants. We assume that $x(t) \geq 2$, one of each gender, for the fish population to survive and $x(0) = x_0 \geq 2$. At every time, the stock x generates αx births and it has $\beta x \ln x$ deaths.

[1] We note that $(1 - s\gamma_-) > 0$.

Two fishermen harvest fish from the lake and each fisherman's catch c_i is directly related to the level of effort $w_i = w_i(t)$ he devotes to this activity and the stock of fish: thus

$$c_i = x_i w_i.$$

Clearly, the fisherman's activity reduced the fish stock in the lake and with respect the equation of the evolution of the fish we have

$$\dot{x} = \alpha x - \beta x \ln x - w_1 x - w_2 x.$$

Each fisherman derives satisfaction from his catch according to a log utility function U_i as

$$U_i = a_i \ln(w_i x),$$

where a_i are positive constants, in an infinite period. Hence we will introduce a discount factor e^{-rt}, with $r > 0$, for such utility.

It is convenient for computations to set $y(t) = \ln x(t)$: hence $\dot{y} = \frac{\dot{x}}{x}$. the target of very player–fisherman is to realize

$$\max_{w_i} \int_0^\infty a_i(y + \ln w_i)e^{-rt}\,\mathrm{d}t = a_i \max_{w_i} \int_0^\infty (y + \ln w_i)e^{-rt}\,\mathrm{d}t,$$

since $a_i > 0$. Hence we have the following symmetric game:[2]

$$\begin{cases} \text{I F.:} \ \max\limits_{w_1} \displaystyle\int_0^\infty (y + \ln w_1)e^{-rt}\,\mathrm{d}t \qquad \text{II F.:} \ \max\limits_{w_2} \displaystyle\int_0^\infty (y + \ln w_2)e^{-rt}\,\mathrm{d}t \\ \qquad w_1 \geq 0 \qquad\qquad\qquad\qquad\qquad w_2 \geq 0 \\ \qquad\qquad \dot{y} = \alpha - w_1 - w_2 - \beta y \\ \qquad y(0) = y_0 \geq \ln 2, \qquad y(t) \geq \ln 2 \end{cases}$$

Let us assume for simplicity

$$\alpha - 3\beta > 2r. \tag{2.39}$$

We are interested on non zero Nash equilibria in the class of open–loop strategies and in the class of feedback strategies.

Open–loop Nash equilibrium.[3] Let us introduce the two current Hamiltonians:

$$H_1^c = y + \ln w_1 + \lambda_{1c}(\alpha - w_1 - w_2 - \beta y)$$
$$H_2^c = y + \ln w_2 + \lambda_{2c}(\alpha - w_1 - w_2 - \beta y)$$

We have to guarantee the following conditions:

$$\nu_i(t, y) \in \arg\max_{v \geq 0} H_i^c = \arg\max_{v \geq 0}(\ln v - \lambda_{ic} v) = \begin{cases} \frac{1}{\lambda_{ic}} & \text{if } \lambda_{ic} > 0 \\ \nexists & \text{if } \lambda_{ic} \leq 0 \end{cases} \tag{2.40}$$

$$\dot{\lambda}_{ic} = r\lambda_{ic} - \frac{\partial H_i^c}{\partial y} = (r + \beta)\lambda_{ic} - 1 \tag{2.41}$$

for $i = 1, 2$. We note that ν_1 and ν_2 in (2.40) do not depend on the trajectory y: hence we are in the position to looking for a open–loop equilibrium. Let us looking for some non zero Nash equilibrium, we obtain by (2.40)

$$w_1(t) = \nu_1(t) = \frac{1}{\lambda_{1c}(t)} \qquad w_2(t) = \nu_2(t) = \frac{1}{\lambda_{2c}(t)} \tag{2.42}$$

[2]Suggestion: In order to solve the Bellman–Hamilton–Jacobi equation for the current value functions, we suggest to looking for the solution in the family of functions

$$V_1^c(y) = ay + b, \qquad V_2^c(y) = cy + d, \tag{2.38}$$

with a, b, c, d constants.

[3]In subsection we looking for a open–loop Stackelberg equilibrium.

in the assumption that

$$\lambda_{ic} > 0. \tag{2.43}$$

The adjoint equations (2.41) give

$$\lambda_{1c}(t) = Ae^{(\beta+r)t} + \frac{1}{\beta+r}, \qquad \lambda_{2c}(t) = Be^{(\beta+r)t} + \frac{1}{\beta+r},$$

with A and B constants. Clearly (2.42) gives, putting in evidence the dependence by the two constants,

$$w_1^A(t) = \frac{\beta+r}{A(\beta+r)e^{(\beta+r)t}+1}, \qquad w_2^B(t) = \frac{\beta+r}{B(\beta+r)e^{(\beta+r)t}+1}. \tag{2.44}$$

The dynamics now gives

$$y^{AB}(t) = e^{-\beta t}\left[\int_0^t \left(\alpha - w_1^A(s) - w_2^B(s)\right) e^{\beta s}\,\mathrm{d}s + y_0\right]. \tag{2.45}$$

We claim that the case $A = B = 0$ is the unique candidate to a be a Nash equilibrium. In order to prove that, first we put $A < 0$ and, for t sufficiently large, we obtain $w_1^A < 0$ which is impossible. The case $B < 0$ is similar. Now, let us suppose that A and B are non negative; note in this case (2.43) hold: we want to prove that

$$\int_0^\infty (y^{AB} + \ln w_1^A)e^{-rt}\,\mathrm{d}t < \int_0^\infty (y^{0B} + \ln w_1^0)e^{-rt}\,\mathrm{d}t, \tag{2.46}$$

i.e. that (w_1^A, w_2^B) is not a Nash equilibrium since for the first player, with w_2^B fixed, there exists a better strategy with respect to w_1^A. Now, taking into account that

$$\begin{aligned}\int_0^t \frac{A(\beta+r)^2e^{(\beta+r)s}}{A(\beta+r)e^{(\beta+r)s}+1}\,\mathrm{d}s &= \\ &= \ln\left(A(\beta+r)e^{(\beta+r)t}+1\right) - \ln(A(\beta+r)+1)\end{aligned} \tag{2.47}$$

for every fixed t we have, by (2.44) and (2.45),

$$\begin{aligned} y^{AB}(t) + \ln w_1^A(t) &= e^{-\beta t}\left[\int_0^t \left(\alpha - \frac{\beta+r}{A(\beta+r)e^{(\beta+r)s}+1} - w_2^B(s)\right) e^{\beta s}\,\mathrm{d}s + y_0\right] + \\ &\quad + \ln(\beta+r) - \ln\left(A(\beta+r)e^{(\beta+r)t}+1\right) \\ \text{(by (2.47))} \quad &= e^{-\beta t}\left[\int_0^t \left(\alpha - w_2^B(s)\right) e^{\beta s}\,\mathrm{d}s + y_0\right] + \ln(\beta+r) + \\ &\quad -(\beta+r)\int_0^t \frac{A(\beta+r)e^{(\beta+r)s} + e^{\beta(s-t)}}{A(\beta+r)e^{(\beta+r)s}+1}\,\mathrm{d}s - \ln\left(A(\beta+r)+1\right) \\ &<^\dagger e^{-\beta t}\left[\int_0^t \left(\alpha - w_2^B(s)\right) e^{\beta s}\,\mathrm{d}s + y_0\right] + \ln(\beta+r) - (\beta+r)\int_0^t e^{\beta(s-t)}\,\mathrm{d}s \\ &= y^{0B}(t) + \ln w_1^0(t), \end{aligned} \tag{2.48}$$

where in the inequality "$<^\dagger$" we use $A(\beta+r) > 0$ and the fact that, for every h and k positive[4] we have,

$$-\frac{k+h}{k+1} \le -h \quad \Leftrightarrow \quad h \le 1.$$

Clearly relation (2.48) implies (2.46).

[4]in our case $h = e^{\beta(s-t)}$

Let us study the case $A = B = 0$, i.e.

$$\begin{aligned} w_1^*(t) &= w_2^*(t) = \beta + r \\ y^*(t) &= y_0 e^{-\beta t} + \frac{\alpha - 2(\beta + r)}{\beta}(1 - e^{-\beta t}) \\ \lambda_{1c}^*(t) &= \lambda_{2c}^*(t) = \frac{1}{\beta + r} \end{aligned} \tag{2.49}$$

Let us check that $y^*(t) \geq \ln 2$: in fact, by plotting the function y^* and by (2.39), we have

$$y^*(t) \geq \min\left(y_0, \frac{\alpha - 2(\beta + r)}{\beta}\right) \geq 1, \qquad t \geq 0.$$

We recall that in order to guarantee some sufficient condition of optimality in a infinite horizon problem (see subsection A.1.1), we require that $\lim_{t\to\infty} \boldsymbol{\lambda}^*(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \geq 0$ where $\boldsymbol{\lambda}^*$ is the multiplier, $\mathbf{x}$ a generic trajectory and $\mathbf{x}^*$ the trajectory candidate to the optimal. Since $\lambda_i^*(t) = e^{-rt}\lambda_{ic}^*(t) = e^{-rt}/(\beta + r)$, this sufficient condition for the i-problem is

$$\lim_{t\to\infty} \lambda_i^*(t)(y(t) - y^*(t)) \geq \lim_{t\to\infty} \frac{1}{\beta + r} e^{-rt}(\ln 2 - y^*(t)) = 0$$

Finally it is easy to see that the Hamiltonians

$$(y, w_1) \mapsto H_1^c(y, w_1, w_2^*(t), \lambda_{1c}^*(t)) \quad \text{and} \quad (y, w_2) \mapsto H_2^c(y, w_1^*(t), w_2, \lambda_{2c}^*(t))$$

are concave functions, for every fixed t, and that the control sets $U_1 = U_2 = [0, \infty)$ are convex. Hence (w_1^*, w_2^*) in (2.49) is a open–loop Nash equilibrium.

Feedback Nash equilibrium. Let us introduce the two current value functions $V_1^c = V_1^c(y)$ and $V_2^c = V_2^c(y)$: the BHJ equations are

$$\begin{cases} -rV_1^c + y + (V_1^c)'(\alpha - \nu_2^* - \beta y) + \max_{v \geq 0} \left[\ln v - v(V_1^c)'\right] = 0 \\ -rV_2^c + y + (V_2^c)'(\alpha - \nu_1^* - \beta y) + \max_{v \geq 0} \left[\ln v - v(V_2^c)'\right] = 0 \end{cases}$$

We obtain that we realize the previous two max for

$$\nu_i^*(t, y) = \nu_i^*(y) = \begin{cases} \dfrac{1}{(V_i^c)'(y)} & \text{if } (V_i^c)'(y) > 0 \\ \nexists & \text{if } (V_i^c)'(y) \leq 0 \end{cases} \tag{2.50}$$

Let us looking for a Nash equilibrium and hence let us suppose $(V_i^c)'(y) > 0$. Hence the BHJ equations become

$$\begin{cases} -rV_1^c + y + (V_1^c)'\left(\alpha - \dfrac{1}{(V_2^c)'} - \beta y\right) - \ln(V_1^c)' - 1 = 0 \\ -rV_2^c + y + (V_2^c)'\left(\alpha - \dfrac{1}{(V_1^c)'} - \beta y\right) - \ln(V_2^c)' - 1 = 0 \end{cases}$$

Now, using the suggestion (2.38), an easy computation in the previous system gives $a = c = \frac{1}{\beta + r}$. Note that the previous assumption $(V_i^c)'(y) > 0$ is true. It is clear, by (2.50), that we obtain

$$\nu_i^*(y) = \beta + r$$

as in (2.49). Since such (w_1^*, w_2^*), with $w_i^*(t) = \nu_i^*(t, y) = \beta + r$, is admissible (solve the dynamics $\dot{y} = \alpha - \nu_1^*(t, y) - \nu_2^*(t, y) - \beta y$ with the initial condition and check that $y^* \geq 2$ with calculations similar to the open–loop case) then it is a feedback Nash–equilibrium.

2.3.2 On international pollution

We denote by $u_i = u_i(t)$, for $i = 1, 2$, the level of emissions of two economies and let $x = x(t)$ be the stock of pollution at time t. The system has a (little) capacity to self-cleaning; let us fix $\alpha \in (0,1)$ such that $\dot{x} = -\alpha x$ describes this capacity of the system.
The damage of the emission is quadratic with respect to x with coefficients $\frac{1}{2}\phi_i$, for every i-player, and we suppose that the utility for the i-economy related to its emission u_i is given by the concave function $u_i\left(k_i - \frac{1}{2}u_i\right)$, with k_i positive constants. Hence we have

$$\begin{cases} \text{I Econ.:} \ \max\limits_{u_1} \displaystyle\int_0^\infty e^{-rt}\left(u_1\left(k_1 - \frac{1}{2}u_1\right) - \frac{1}{2}\phi_1 x^2\right) \mathrm{d}t \qquad u_1 \geq 0 \\ \text{II Econ.:} \ \max\limits_{u_2} \displaystyle\int_0^\infty e^{-rt}\left(u_2\left(k_2 - \frac{1}{2}u_2\right) - \frac{1}{2}\phi_2 x^2\right) \mathrm{d}t \qquad u_2 \geq 0 \\ \dot{x} = u_1 + u_2 - \alpha x \\ x(0) = x_0, \qquad x(t) \geq 0 \end{cases}$$

with the rate of discount r that is a positive constant. This model is proposed in [5].
Open loop Nash equilibrium, in the general case. Let us introduce the two current Hamiltonians:

$$H_1^c = k_1 u_1 - \frac{1}{2}u_1^2 - \frac{1}{2}\phi_1 x^2 + \lambda_{1c}(u_1 + u_2 - \alpha x)$$
$$H_2^c = k_2 u_2 - \frac{1}{2}u_2^2 - \frac{1}{2}\phi_2 x^2 + \lambda_{2c}(u_1 + u_2 - \alpha x)$$

We have to guarantee the following conditions:

$$\nu_i(t,x) \in \arg\max_{v \geq 0} H_i^c = \arg\max_{v\geq 0}\left((k_i + \lambda_{ic})v - \frac{1}{2}v^2\right) = \\ = \begin{cases} k_i + \lambda_{ic} & \text{if } k_i + \lambda_{ic} \geq 0 \\ 0 & \text{if } k_i + \lambda_{ic} < 0 \end{cases} \tag{2.51}$$

$$\dot{\lambda}_{ic} = r\lambda_{ic} - \frac{\partial H_i^c}{\partial x} = (\alpha + r)\lambda_{ic} + \phi_i x \tag{2.52}$$

for $i = 1, 2$. We note that ν_1 and ν_2 in (2.51) do not depend on the trajectory x: hence we are in the position to looking for a open–loop equilibrium.

Let us looking for some non zero Nash equilibrium: more precisely we are interested in the case where the emissions u_i are positive in $[0, \infty)$, i.e. we assume that in (2.51) we have

$$k_i + \lambda_{ic}(t) > 0, \qquad \forall t \tag{2.53}$$

Taking into account (2.51) in the dynamics and with (2.52), we have to solve the system

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda}_{1c} \\ \dot{\lambda}_{2c} \end{pmatrix} = A \begin{pmatrix} x \\ \lambda_{1c} \\ \lambda_{2c} \end{pmatrix} + \begin{pmatrix} k_1 + k_2 \\ 0 \\ 0 \end{pmatrix}, \text{ with } A = \begin{pmatrix} -\alpha & 1 & 1 \\ \phi_1 & \alpha + r & 0 \\ \phi_2 & 0 & \alpha + r \end{pmatrix}$$

The eigenvalues of A are $\theta = \alpha + r$ and

$$\theta_{\pm} = \frac{r \pm \sqrt{r^2 + 4(\alpha^2 + \alpha r + \phi_1 + \phi_2)}}{2}. \tag{2.54}$$

Three eigenvectors for each eigenvalue are $v = (0,1,-1)^T$ (related to θ) and $v_\pm = (\alpha + r - \theta_\pm, -\phi_1, -\phi_2)^T$ respectively. Hence the general solution for the homogeneous part of our system is given by

$$(x, \lambda_{1c}, \lambda_{2c})^T = c_1 e^{t\theta} v + c_2 e^{t\theta_+} v_+ + c_3 e^{t\theta_-} v_-$$

where c_i are generic constants. It is easy to see that a particular solution of our system is given by

$$(x^{part}, \lambda_{1c}^{part}, \lambda_{2c}^{part})^T = \frac{(k_1 + k_2)}{\alpha(\alpha + r) + \phi_1 + \phi_2} \left((\alpha + r),\ -\phi_1,\ -\phi_2\right)^T$$

Hence we obtain

$$\begin{aligned} x(t) &= c_2 e^{t\theta_+}(\alpha + r - \theta_+) + c_3 e^{t\theta_-}(\alpha + r - \theta_-) + x^{part} \\ \lambda_{1c}(t) &= c_1 e^{(\alpha+r)t} - \phi_1 \left(c_2 e^{t\theta_+} + c_3 e^{t\theta_-}\right) + \lambda_{1c}^{part} \\ \lambda_{2c}(t) &= -c_1 e^{(\alpha+r)t} - \phi_2 \left(c_2 e^{t\theta_+} + c_3 e^{t\theta_-}\right) + \lambda_{2c}^{part} \end{aligned} \tag{2.55}$$

Using the expression of λ_{1c} given by (2.55), putting (see (2.51))

$$u_1(t) = \nu_1(t) = k_1 + \lambda_{1c}(t),$$

and taking into account that

$$2\theta - r > 0 \qquad 2\theta_+ - r > 0,$$

it is easy to see that, if $(c_1, c_2) \neq (0,0)$, then for $t \to \infty$ we have

$$e^{-rt}\left(u_1\left(k_1 - \frac{1}{2}u_1\right) - \frac{1}{2}\psi_1 x^2\right) \sim -\frac{1}{2}e^{-rt}\left((\lambda_{1c})^2 + \psi_1 x^2\right) \to -\infty$$

Hence we have $c_1 = c_2 = 0$ and, by the initial condition on the stock of pollution

$$\begin{aligned} x^*(t) &= \left(x_0 - \frac{(\alpha + r)(k_1 + k_2)}{\alpha(\alpha + r) + \phi_1 + \phi_2}\right) e^{t\theta_-} + \frac{(\alpha + r)(k_1 + k_2)}{\alpha(\alpha + r) + \phi_1 + \phi_2} \\ \lambda_{1c}^*(t) &= -\frac{\phi_1}{\alpha + r - \theta_-}\left(x_0 - \frac{(\alpha + r)(k_1 + k_2)}{\alpha(\alpha + r) + \phi_1 + \phi_2}\right) e^{t\theta_-} - \frac{(k_1 + k_2)\phi_1}{\alpha(\alpha + r) + \phi_1 + \phi_2} \\ \lambda_{2c}^*(t) &= -\frac{\phi_2}{\alpha + r - \theta_-}\left(x_0 - \frac{(\alpha + r)(k_1 + k_2)}{\alpha(\alpha + r) + \phi_1 + \phi_2}\right) e^{t\theta_-} - \frac{(k_1 + k_2)\phi_2}{\alpha(\alpha + r) + \phi_1 + \phi_2} \end{aligned}$$

Note that $x^*(t) > 0$. We have to verify that (2.53) holds. In order to do that note that

$$\dot{\lambda}_{ic}^*(t) > 0 \qquad \Leftrightarrow \qquad x_0 > x^{part}$$

Hence, in the case $x_0 \geq x^{part}$ we have to check that $\lambda_{ic}^*(0) + k_i > 0$, while in the case $x_0 < x^{part}$ we have to check that $\lambda_{ic}^{part} + k_i > 0$. We are not interested on this tedious calculations.

Let us prove that (u_1^*, u_2^*) defined by (2.51)

$$u_1^*(t) = \nu_1^*(t) = k_1 + \lambda_{1c}^*(t), \qquad u_2^*(t) = \nu_2^*(t) = k_2 + \lambda_{2c}^*(t),$$

and using λ_{ic}^* as in the last expression. Let us study the situation from the point of view of the first player. It is immediate to see that

$$(x, u_1) \mapsto H_1^c(t, x, u_1, u_2^*(t), \lambda_{1c}^*(t))$$

is, for every fixed t, a concave function. It is clear that for every admissible trajectory $x = x(t)$ we have, in order to have $x^2 e^{-rt}$ integrable for $t \to \infty$,

$$x^2(t)e^{-rt} \to 0, \quad t \to \infty.$$

Note that this is equivalent $x(t)e^{-rt/2} \to 0$, for $t \to \infty$. Recalling that $\lambda_1^*(t) = e^{-rt}\lambda_{1c}^*(t)$, we have

$$\lim_{t\to\infty} \lambda_i^*(t)(x(t) - x^*(t)) = \lambda_{ic}^{part} \lim_{t\to\infty} e^{-rt}(x(t) - x^{part}) = 0$$

Hence u_1^* is optimal for the first player (see subsection A.1.1): similar arguments hold for the second economy and hence we have really that (u_1^*, u_2^*), defined by (2.51), is a Nash equilibrium in the class of open loop strategies.

Chapter 3

Stackelberg equilibria

In 1934, von Stackelberg introduced a concept of a hierarchical solution for markets where some firms have power of domination over others. This solution concept is now known as the Stackelberg equilibrium or the Stackelberg solution which, in the context of two-persons nonzero-sum games, involves players with asymmetric roles, one leading (accordingly called the Leader) and the other one following (called the Follower).

In all this chapter we are considering the following hierarchy two–person game

$$\begin{cases} \text{Player I (Leader): } \max_{\mathbf{u}_L} J_L(\mathbf{u}_L, \mathbf{u}_F) \\ J_L(\mathbf{u}_L, \mathbf{u}_F) = \int_0^T f_L(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F)\, dt + \psi_L(\mathbf{x}(T)) \\ \text{Player II (Follower): } \max_{\mathbf{u}_F} J_F(\mathbf{u}_L, \mathbf{u}_F) \\ J_F(\mathbf{u}_L, \mathbf{u}_F) = \int_0^T f_F(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F)\, dt + \psi_F(\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{3.1}$$

where T is fixed and $(\mathbf{u}_L, \mathbf{u}_F)$ is an admissible control, i.e. depending on the information structure. We assume that the control sets for the Leader and for the Follower are U_L and U_F respectively, closed.

3.1 Open-loop Stackelberg equilibria

Let us consider a open–loop strategy $\mathbf{u}_L$, with $\mathbf{u}_L(t) = \boldsymbol{\nu}_L(t, \mathbf{x}_0)$, for the Leader. We define **the set of best possible replies** of the Follower $\mathcal{R}^F(\mathbf{u}_L)$ in the family of the open–loop strategies, where the Leader has already announced the strategy $\mathbf{u}_L$, as

$$\mathcal{R}^F(\mathbf{u}_L) = \Big\{\mathbf{u}_F : (\mathbf{u}_L, \mathbf{u}_F) \in \mathcal{A}_{OL}, \quad J_F(\mathbf{u}_L, \mathbf{u}'_F) \le J_F(\mathbf{u}_L, \mathbf{u}_F) \ \ \forall (\mathbf{u}_L, \mathbf{u}'_F) \in \mathcal{A}_{OL}\Big\}.$$

Clearly $(\mathbf{u}_L, \mathbf{u}_F) \in \mathcal{R}^F(\mathbf{u}_L)$ is an admissible pair of strategies and the set $\mathcal{R}^F(\mathbf{u}_L)$ can be empty.

Definition 3.1. *A pair of control functions $(\mathbf{u}^*_L, \mathbf{u}^*_F) \in \mathcal{A}_{OL}$, with $\mathbf{u}^*_L(t) = \boldsymbol{\nu}^*_L(t, \mathbf{x}_0)$ and $\mathbf{u}^*_F(t) = \boldsymbol{\nu}^*_F(t, \mathbf{x}_0)$, is a* ***Stackelberg equilibrium within the class of open–loop strategies*** *$\mathcal{A}_{OL}$ if*

i. $\mathbf{u}^*_F \in \mathcal{R}^F(\mathbf{u}^*_L)$, *with associated trajectory* $\mathbf{x}^*$*;*

ii. given any open–loop strategy $\mathbf{u}_L$ for the Leader and every best reply $\mathbf{u}_F \in \mathcal{R}^F(\mathbf{u}_L)$ for the Follower, the following holds

$$\int_0^T f_L(t, \mathbf{x}_{LF}, \mathbf{u}_L, \mathbf{u}_F)\ dt + \psi_L(\mathbf{x}_{LF}(T)) \le$$

$$\le \int_0^T f_L(t, \mathbf{x}^*, \mathbf{u}_L^*, \mathbf{u}_F^*)\ dt + \psi_L(\mathbf{x}^*(T))$$

where $\mathbf{x}_{LF}$ *is the trajectory associated to the pair* $(\mathbf{u}_L, \mathbf{u}_F)$.

In this case the problem has to be solved backward.

First of all, let us assume that

As1 the function f_L, f_F, g, ψ_L are continuously differentiable w.r.t. $\mathbf{x}$.

From the point of view of the Follower, for every $\mathbf{u}_L$ fixed, we construct the set of best possible replies $\mathcal{R}^F(\mathbf{u}_L)$. We have to solve, for a fixed $\mathbf{u}_L$

$$\begin{cases} \max\limits_{\mathbf{u}_F} \int_0^T f_F\,(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F)\, dt + \psi_F(\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{3.2}$$

Applying Theorem A.1, if $\mathbf{u}_F \in \mathcal{R}^F(\mathbf{u}_L)$ and $\mathbf{x}$ is the trajectory associated to $(\mathbf{u}_L, \mathbf{u}_F)$, then there exists a continuous multiplier[1] $\boldsymbol{\lambda}_F : [0,T] \to \mathbb{R}^n$ such that

$$\boldsymbol{\nu}_F(t, \mathbf{x}_0) \in \arg\max_{\mathbf{v}\in U_F} H_F(t, \mathbf{x}(t), \mathbf{u}_L(t), \mathbf{v}, \boldsymbol{\lambda}_F(t)) \qquad \forall t \in [0,T] \tag{3.3}$$

$$\dot{\boldsymbol{\lambda}}_F = -\nabla_{\mathbf{x}} H_F(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F, \boldsymbol{\lambda}_F) \qquad \text{in } [0,T] \tag{3.4}$$

$$\boldsymbol{\lambda}_F(T) = \nabla\psi_F(\mathbf{x}(T)) \tag{3.5}$$

where $\mathbf{u}_F(t) = \boldsymbol{\nu}_F(t, \mathbf{x}_0)$ and the Hamiltonian H_F of the Follower is defined by

$$H_F(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F, \boldsymbol{\lambda}_F) = f_F(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F) + \boldsymbol{\lambda}_F \cdot g(t, \mathbf{x}, \mathbf{u}_L, \mathbf{u}_F). \tag{3.6}$$

We note that $\mathbf{x}$ depends on the choice of $\mathbf{u}_L$ and on the choice of $\mathbf{u}_F$ in $\mathcal{R}^F(\mathbf{u}_L)$ (with the possibility that this second choice can be not unique): hence $\mathbf{x}$ depends on $(t, \mathbf{u}_L, \mathbf{u}_F)$, i.e. $\mathbf{x} = \mathbf{x}(t, \mathbf{u}_L, \mathbf{u}_F)$. Moreover, the multiplier is associated to the pair trajectory–control functions $(\mathbf{x}, (\mathbf{u}_L, \mathbf{u}_F))$: hence $\boldsymbol{\lambda}_F$ depends on $(\mathbf{x}, \mathbf{u}_L, \mathbf{u}_F)$, i.e. $\boldsymbol{\lambda}_F = \boldsymbol{\lambda}_F(\mathbf{x}, \mathbf{u}_L, \mathbf{u}_F)$.
We are looking for a open–loop strategy: hence let us assume that

As2 for every $(t, \mathbf{x}_0, \mathbf{x}, \mathbf{u}_L, \boldsymbol{\lambda}_F)$ there exists a unique max in

$$\boldsymbol{\nu}_F \in \arg\max_{\mathbf{v}\in U_F} H_F(t, \mathbf{x}, \mathbf{u}_L, \mathbf{v}, \boldsymbol{\lambda}_F)$$

which does not depend on $\mathbf{x}$.

Taking into account of the previous dependence, we have $\boldsymbol{\nu}_F = \boldsymbol{\nu}_F(t, \mathbf{x}_0, \mathbf{u}_L, \boldsymbol{\lambda}_F)$. Moreover, since in this situation the value $\mathbf{x}(T)$ of the trajectory at the final fixed time T is free, then our controls are normal: this is the reason of our definition of H_F in (3.6). At this point we have to guarantee some sufficient conditions for the game of the Follower.

Now, let us consider the point of view of the Leader. For every $\mathbf{u}_L$ we associated $(\mathbf{u}_F, \mathbf{x}, \boldsymbol{\lambda}_F)$ where $\mathbf{u}_F \in \mathcal{R}^F(\mathbf{u}_L)$ is given by (3.3) and by $\mathbf{u}_F(t) = \boldsymbol{\nu}_F(t, \mathbf{x}_0, \mathbf{u}_L(t), \boldsymbol{\lambda}_F(t))$; $\mathbf{x}$ is given by the dynamics of the problem and $\boldsymbol{\lambda}_L$ is given by (3.4): in this procedure, the best choice for the Leader is $\mathbf{u}_L^*$. Hence the Leader has to solve the following problem, where its control is $\mathbf{u}_L$ and its trajectory is $(\mathbf{x}, \boldsymbol{\lambda}_F)$ with the conditions $\mathbf{x}(0) = \mathbf{x}_0$ and (3.5):

$$\begin{cases} \max\limits_{\mathbf{u}_L} \int_0^T f_L\,(t, \mathbf{x}, \mathbf{u}_L, \boldsymbol{\nu}_F(t, \mathbf{x}_0, \mathbf{u}_L, \boldsymbol{\lambda}_F))\, dt + \psi_L(\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_L, \boldsymbol{\nu}_F(t, \mathbf{x}_0, \mathbf{u}_L, \boldsymbol{\lambda}_F)) \\ \dot{\boldsymbol{\lambda}}_F = -\nabla_{\mathbf{x}} H_F(t, \mathbf{x}, \mathbf{u}_L, \boldsymbol{\nu}_F(\mathbf{u}_L, \boldsymbol{\lambda}_F), \boldsymbol{\lambda}_F) \\ \mathbf{x}(0) = \mathbf{x}_0 \\ \boldsymbol{\lambda}_F(T) = \nabla\psi_F(\mathbf{x}(T)) \end{cases} \tag{3.7}$$

[1] We omit all the "*".

In this case the Hamiltonian H_L for the Leader is defined by

$$\begin{aligned} H_L(t,\mathbf{x},\boldsymbol{\lambda}_F,\mathbf{u}_L,\lambda_{0L},\boldsymbol{\lambda}_{1L},\boldsymbol{\lambda}_{2L}) &= \lambda_{0L} f_L(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) + \\ &\quad +\boldsymbol{\lambda}_{1L}\cdot g(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) + \\ &\quad -\boldsymbol{\lambda}_{2L}\cdot \nabla_{\mathbf{x}} H_F(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F),\boldsymbol{\lambda}_F) \end{aligned} \tag{3.8}$$

We note that in the definition of H_L, since the value of the trajectory $(\mathbf{x},\boldsymbol{\lambda}_F)$ has a condition in the initial point $t=0$ and in the final point $t=T$ (more precisely the final equation in (3.7) represents a surface), we are not in the position to guarantee the normality of the extremal: hence we insert λ_0.
In order to apply again Theorem A.1, let us assume that (taking into account (3.6))

As3 for every $(t,\mathbf{u}_L)$ the functions

$$\begin{aligned} &(\mathbf{x},\boldsymbol{\lambda}_F)\mapsto f_L(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) \\ &(\mathbf{x},\boldsymbol{\lambda}_F)\mapsto g(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) \\ &(\mathbf{x},\boldsymbol{\lambda}_F)\mapsto f_F(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) + \boldsymbol{\lambda}_F\cdot g(t,\mathbf{x},\mathbf{u}_L,\boldsymbol{\nu}_F(t,\mathbf{x}_0,\mathbf{u}_L,\boldsymbol{\lambda}_F)) \end{aligned}$$

are in C^1.

Hence if $\mathbf{u}_L^*$ is a Stackelberg equilibrium, there then there exists a continuous multiplier[2] $(\lambda_{0L},\boldsymbol{\lambda}_{1L},\boldsymbol{\lambda}_{2L}) : [0,T]\to\mathbb{R}^{2n+1}$ such that λ_{0L} is a non negative constant, $(\lambda_{0L},\boldsymbol{\lambda}_{1L},\boldsymbol{\lambda}_{2L})\neq(0,0,0)$ and

$$\begin{aligned} &\boldsymbol{\nu}_L(t,\mathbf{x}_0)\in\arg\max_{\mathbf{v}\in U_L} H_L(t,\mathbf{x}(t),\boldsymbol{\lambda}_F(t),\mathbf{v},\lambda_{0L},\boldsymbol{\lambda}_{1L}(t),\boldsymbol{\lambda}_{2L}(t)) \quad \forall t\in[0,T] \\ &\dot{\boldsymbol{\lambda}}_{1L} = -\nabla_{\mathbf{x}} H_L(t,\mathbf{x},\boldsymbol{\lambda}_F,\mathbf{u}_L,\lambda_{0L},\boldsymbol{\lambda}_{1L},\boldsymbol{\lambda}_{2L}) \qquad \text{in } [0,T] \\ &\dot{\boldsymbol{\lambda}}_{2L} = -\nabla_{\boldsymbol{\lambda}_F} H_L(t,\mathbf{x},\boldsymbol{\lambda}_F,\mathbf{u}_L,\lambda_{0L},\boldsymbol{\lambda}_{1L},\boldsymbol{\lambda}_{2L}) \qquad \text{in } [0,T] \\ &\boldsymbol{\lambda}_{1L}(T) = \lambda_{0L}\nabla\psi_L(\mathbf{x}(T)) - \lambda_{2L}(T) D^2\psi_F(\mathbf{x}(T)) \\ &\boldsymbol{\lambda}_{2L}(0) = 0 \end{aligned} \tag{3.9}$$

with $\mathbf{u}_L(t)=\boldsymbol{\nu}_L(t,\mathbf{x}_0)$ since we are looking for a open–loop strategy. In (3.9) $D^2\psi_F(x(T))$ denotes the Hessian matrix of second derivatives of ψ_F, evaluated in $x(T)$; such transversality condition (3.9) is a consequence of the mentioned surface (3.5): see section 4.2 in [3] for all details.

Clear, up to now we are discussing only of sufficient conditions of optimality for the two problems (3.2) and (3.7): in order to find a Stackelberg equilibrium for (3.1) we have to guarantee some sufficient conditions for the two mentioned previous problems.

3.1.1 On international pollution with hierarchical relations

Let us denote by $u_L=u_L(t)$ and $u_F=u_F(t)$ the level of emissions of two economies, where the first (the Leader) has a sort of domination with respect to the second economy (the Follower); for example, this is the situation that occurs when the Follower has a big debit with the Leader. Now, as in subsection 2.3.2, let $x=x(t)$ be the stock of pollution at time t, $\alpha\in(0,1)$ is the coefficient of capacity to self-cleaning of the system. The damage of the emission is quadratic with respect to x with coefficients $\frac{1}{2}\phi_L$ and $\frac{1}{2}\phi_F$, and we suppose that the utility for the the two economies related to its emission are quadratic. As in subsection 2.3.2, we have

$$\begin{cases} \text{Leader:} \ \max\limits_{u_L}\displaystyle\int_0^\infty e^{-rt}\left(u_L\left(k_L-\frac{1}{2}u_L\right)-\frac{1}{2}\phi_L x^2\right)\,\mathrm{d}t \qquad u_L\geq 0 \\ \text{Follower:} \ \max\limits_{u_F}\displaystyle\int_0^\infty e^{-rt}\left(u_F\left(k_F-\frac{1}{2}u_F\right)-\frac{1}{2}\phi_F x^2\right)\,\mathrm{d}t \qquad u_F\geq 0 \\ \dot{x}=u_L+u_F-\alpha x \\ x(0)=x_0, \qquad x(t)\geq 0 \end{cases}$$

[2] We omit again all the "*".

with the rate of discount r that is a positive constant and with ϕ_L, ϕ_F, K_L and K_F positive constants. Let us looking for a non zero Stackelberg equilibrium in the family of open–loop strategies.

Let us introduce the current Hamiltonian H_F^c for the Follower

$$H_F^c = k_F u_F - \frac{1}{2}u_F^2 - \frac{1}{2}\phi_F x^2 + \lambda_{Fc}(u_L + u_F - \alpha x).$$

Let us fix u_L: hence all that follows for the Follower depends on such u_L. We have to guarantee the following conditions:

$$\nu_F \in \arg\max_{u_F \geq 0} H_F^c = \arg\max_{v \geq 0} \left((k_F + \lambda_{Fc})v - \frac{1}{2}v^2 \right) = \\ = \begin{cases} k_F + \lambda_{Fc} & \text{if } k_F + \lambda_{Fc} \geq 0 \\ 0 & \text{if } k_F + \lambda_{Fc} < 0 \end{cases} \tag{3.10}$$

$$\dot{\lambda}_{Fc} = r\lambda_{Fc} - \frac{\partial H_F^c}{\partial x} = (\alpha + r)\lambda_{Fc} + \phi_F x \tag{3.11}$$

where $\nu_F = \nu_F(t, x, u_L)$, $\lambda_{Fc} = \lambda_{Fc}(t, u_L)$ and $x = x(t, u_L)$. We note that ν_F in (3.10) does not depend on the trajectory x: hence we are in the position to looking for a open–loop equilibrium.

Let us looking for some Stackelberg equilibrium where the emissions are positive in $[0, \infty)$, i.e. we assume that in (3.10) we have

$$k_F + \lambda_{Fc}(t, u_L) > 0, \qquad \forall t \tag{3.12}$$

We add to these necessary conditions (3.10)–(3.11), some considerations with respect to the sufficient conditions of optimality for the Follower, for every u_L fixed by the Leader. Suppose that for every fixed u_L we have a extremal tern $(x_F^*(t, u_L), \lambda_{Fc}^*(t, u_L), u_F^*(t, u_L))$: it is immediate to see that, always for every fixed u_L,

$$(x, u_F) \mapsto H_F^c(t, x, u_L(t), u_F, \lambda_{Fc}^*(t, u_L))$$

is, for every fixed t, a concave function. Moreover we have to guarantee that for every admissible trajectory $x = x(t, u_L)$ we have

$$\lim_{t \to \infty} \lambda_{Fc}^*(t, u_F) e^{-rt}(x(t, u_L) - x^*(t, u_L)) \geq 0. \tag{3.13}$$

In order to do that, let us note that if $x(t, u_L)$ is associated to an admissible control, then $-\frac{\phi_F}{2}(x(t, u_L))^2 e^{-rt} \to 0$ for $t \to \infty$, and hence

$$\lim_{t \to \infty} e^{-rt/2} x(t, u_L) = 0 \tag{3.14}$$

Now, if $|\lambda_{Fc}^*(t, u_F)| \to \infty$ for $t \to \infty$, then $|u^*(t, u_F)| = |k_F + \lambda_{Fc}^*(t, u_F)| \to \infty$. Again, if $u^*(t, u_F)$ is admissible, then $-\frac{1}{2}(u^*(t, u_L))^2 e^{-rt} \to 0$ for $t \to \infty$, and hence

$$\lim_{t \to \infty} u^*(t, u_L) e^{-rt/2} = \lim_{t \to \infty} \lambda_{Fc}^*(t, u_F)^{-rt/2} = 0. \tag{3.15}$$

Clearly, (3.14) and (3.15) imply (3.13).

Let us pass to the point of view of the Leader; its current Hamiltonian H_L^c is, taking into account (3.10), (3.11) and (3.12),

$$\begin{aligned} H_L^c \quad = \quad & \lambda_0 \left(k_L u_L - \frac{1}{2}u_L^2 - \frac{1}{2}\phi_L x^2 \right) + \lambda_{1Lc}(u_L + K_F + \lambda_{Fc} - \alpha x) + \\ & + \lambda_{2Lc}\left((\alpha + r)\lambda_{Fc} + \phi_F x \right). \end{aligned}$$

We note that we have no conditions on the final point $t = \infty$ of the trajectory $t \mapsto (x(t), \lambda_{Fc}(t))$: hence we are in the position to put $\lambda_0 = 1$. Now we have to guarantee the following necessary conditions:

$$\nu_L \in \arg\max_{u_L \geq 0} H_L^c = \arg\max_{v \geq 0} \left((k_F + \lambda_{1Lc} +)v - \frac{1}{2}v^2 \right)$$

$$= \begin{cases} k_L + \lambda_{1Lc} & \text{if } k_L + \lambda_{1Lc} \geq 0 \\ 0 & \text{if } k_L + \lambda_{1Lc} < 0 \end{cases} \tag{3.16}$$

$$\dot{\lambda}_{1Lc} = r\lambda_{1Lc} - \frac{\partial H_L^c}{\partial x} = (\alpha + r)\lambda_{1Lc} + \phi_L x - \phi_F \lambda_{2Lc} \tag{3.17}$$

$$\dot{\lambda}_{2Lc} = r\lambda_{2Lc} - \frac{\partial H_L^c}{\partial \lambda_{Fc}} = -\lambda_{1Lc} - \alpha\lambda_{2Lc} \tag{3.18}$$

$$\lambda_{2Lc}(0) = 0 \tag{3.19}$$

Since that ν_L in (3.16) does not depend on the trajectory x, we are in the position to looking for a open–loop equilibrium. Let us assume

$$k_L + \lambda_{1Lc}(t) > 0, \qquad \forall t \tag{3.20}$$

Hence, by (3.10) and (3.16), we have $u_F = K_F + \lambda_{Fc}$ and $u_L = K_L + \lambda_{1Lc}$. Putting these information in the dynamic, together with (3.11), (3.17), (3.18), we have to solve the system

$$\dot{\mathbf{z}} = A\mathbf{z} + \mathbf{z}_0 \tag{3.21}$$

where

$$\mathbf{z} = \begin{pmatrix} x \\ \lambda_{Fc} \\ \lambda_{1Lc} \\ \lambda_{2Lc} \end{pmatrix}, \quad \mathbf{z}_0 = \begin{pmatrix} k_1 + k_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -\alpha & 1 & 1 & 0 \\ \phi_F & \alpha + r & 0 & 0 \\ \phi_L & 0 & \alpha + r & -\phi_F \\ 0 & 0 & -1 & -\alpha \end{pmatrix}$$

with the conditions

$$x(0) = x_0, \qquad \lambda_{2Lc}(0) = 0 \tag{3.22}$$

The eigenvalues θ of A solve $\det(A - I\theta) = 0$, i.e. using the first line for the computation of the determinant

$$[(-\alpha - \theta)(r + \alpha - \theta) - \phi_F]^2 - (-\alpha - \theta)(r + \alpha - \theta)\phi_L = 0.$$

Setting

$$A = (-\alpha - \theta)(r + \alpha - \theta) \tag{3.23}$$

we obtain $A^2 - A(2\phi_F + \phi_L) + \phi_F^2 = 0$. Hence

$$A_\pm = \frac{2\phi_F + \phi_L \pm \sqrt{4\phi_L\phi_F + \phi_L^2}}{2}. \tag{3.24}$$

Note that $A_\pm > 0$. Putting $A_\pm$ in (3.23) and solving we obtain

$$\theta_{\pm\pm} = \frac{r \pm \sqrt{r^2 + 4(\alpha^2 + \alpha r + A_\pm)}}{2}. \tag{3.25}$$

where the first "$\pm$" in the subscript of θ is related to the $\pm$ in front of "$\surd$" and the second is related to $A_\pm$. Let $\mathbf{z}^{part} = (x^{part}, \lambda_{Fc}^{part}, \lambda_{1Lc}^{part}, \lambda_{2Lc}^{part})^T \neq \mathbf{0}$ be such that $A\mathbf{z}^{part} = \mathbf{0}$; clear this is a particular solution of (3.21). Hence the general solution of (3.21) is

$$\mathbf{z}(t) = c_1 e^{t\theta_{++}} v_{++} + c_2 e^{t\theta_{+-}} v_{+-} + c_3 e^{t\theta_{-+}} v_{-+} + c_4 e^{t\theta_{--}} v_{--} + \mathbf{z}^{part} \tag{3.26}$$

where c_i are generic constants and $v_{\pm\pm}$ eigenvectors relative to the eigenvalue $\theta_{\pm\pm}$.

We claim that $c_1 = c_2 = 0$: in order to prove that, let us denote by $\mathbf{v}_{++} \in \mathbb{R}^4$ the eigenvector related to the eigenvalue θ_{++}. Let us suppose that the first component v^1_{++} of $\mathbf{v}_{++}$ is zero. It is easy to see that $(A - I\theta_{++})\mathbf{v}_{++} = \mathbf{0}$ implies $\mathbf{v}_{++} = \mathbf{0}$. Hence $v^1_{++} \neq 0$.
Now, if $c_1 \neq 0$, (3.26) implies that, for $t \to \infty$,

$$x(t)^2 e^{-rt} \sim \left(c_1 v^1_{++}\right)^2 e^{(2\theta_{++}-r)t} \to \infty$$

and integral, in our model, does not converge. Hence $c_1 = 0$. Similar arguments imply that $c_2 = 0$. Hence we obtain

$$\begin{aligned} \mathbf{z}(t) &= (x^*(t), \lambda^*_{Fc}(t), \lambda^*_{1Lc}(t), \lambda^*_{2Lc}(t))^T \\ &= c_3 e^{t\theta_{-+}} v_{-+} + c_4 e^{t\theta_{--}} v_{--} + \mathbf{z}^{part} \end{aligned} \tag{3.27}$$

where the two constants c_3 and c_4 depends on the two initial conditions (3.22). We are not interested to discuss the sufficient conditions for the Leader.

3.1.2 Father and son, fishermens at the lake

The model is presented in 2.3.1, but now the situation for the two players is hierarchical. The first player, the father, decide to use its influence on the second player, the son. Let us looking for a Stackelberg equilibrium in the family of open–loop strategies. Let us rewrite the problem taking into account that the the father is the Leader and the son is the Follower:

$$\begin{cases} \text{Leader (father):} \ \max\limits_{w_L} \displaystyle\int_0^\infty (y + \ln w_L) e^{-rt}\, dt \qquad \text{Follower (son):} \ \max\limits_{w_F} \displaystyle\int_0^\infty (y + \ln w_F) e^{-rt}\, dt \\ w_L \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad w_F \geq 0 \\ \dot{y} = \alpha - w_L - w_F - \beta y \\ y(0) = y_0 \geq \ln 2, \qquad y(t) \geq \ln 2 \end{cases}$$

Let us assume for simplicity

$$\alpha - 3\beta > 2r. \tag{3.28}$$

Let us fix w_L the strategy of the father. We consider **the point of view of the Follower–son** and we look for $\mathcal{R}^F(w_L)$, the set of best replies for the son: the current Hamiltonian is

$$H^c_F = y + \ln w_F + \lambda_{Fc}(\alpha - w_L - w_F - \beta y).$$

We have to guarantee the following conditions:

$$\nu_F(t) \in \arg\max_{v \geq 0} H^c_F = \arg\max_{v\geq 0}(\ln v - v\lambda_{Fc}(t)) = \begin{cases} \frac{1}{\lambda_{Fc}} & \text{if } \lambda_{Fc}(t) > 0 \\ \nexists & \text{if } \lambda_{Fc}(t) \leq 0 \end{cases} \tag{3.29}$$

$$\dot{\lambda}_{Fc} = r\lambda_{Fc} - \frac{\partial H^c_F}{\partial y} = (r + \beta)\lambda_{Fc} - 1 \tag{3.30}$$

Note that ν_F in (3.29) do not depend on the trajectory y: hence we are in the position to looking for a open–loop equilibrium

$$w_F(t) = \nu_F(t) = \frac{1}{\lambda_{Fc}(t)} \tag{3.31}$$

in the assumption that

$$\lambda_{Fc}(t) > 0. \tag{3.32}$$

The adjoint equation (3.30) gives

$$\lambda_{Fc}(t) = Be^{(\beta+r)t} + \frac{1}{\beta + r},$$

with B constants. Clearly (3.31) gives, putting in evidence the dependence by B,

$$w_F^B(t) = \frac{\beta + r}{B(\beta + r)e^{(\beta+r)t} + 1} \tag{3.33}$$

and the dynamics implies

$$y^B(t) = e^{-\beta t}\left[\int_0^t \left(\alpha - w_L(s) - w_F^B(s)\right) e^{\beta s}\,\mathrm{d}s + y_0\right]. \tag{3.34}$$

We claim that the case $B = 0$ is the unique candidate to be in $\mathcal{R}^F(w_L)$. In order to prove that, first we put $B < 0$ and, for t sufficiently large, we obtain $w_F^B < 0$ which is impossible. Now, let us suppose that $B \geq 0$: note that in this case (3.32) holds: we want to prove that

$$\int_0^\infty (y^B + \ln w_F^B)e^{-rt}\,\mathrm{d}t < \int_0^\infty (y^0 + \ln w_F^0)e^{-rt}\,\mathrm{d}t, \tag{3.35}$$

i.e. $w_F^B \notin \mathcal{R}^F(w_L)$ for $B > 0$. Now, taking into account that

$$\begin{aligned}\int_0^t \frac{B(\beta+r)^2 e^{(\beta+r)s}}{B(\beta+r)e^{(\beta+r)s}+1}\,\mathrm{d}s &= \\ &= \ln\left(B(\beta+r)e^{(\beta+r)t}+1\right) - \ln(B(\beta+r)+1)\end{aligned} \tag{3.36}$$

for every fixed t we have, by (3.33) and (3.34),

$$\begin{aligned} y^B(t) + \ln w_F^B(t) &= e^{-\beta t}\left[\int_0^t \left(\alpha - w_L(s) - \frac{\beta+r}{B(\beta+r)e^{(\beta+r)s}+1}\right)e^{\beta s}\,\mathrm{d}s + y_0\right] + \\ &\quad + \ln(\beta+r) - \ln\left(B(\beta+r)e^{(\beta+r)t}+1\right) \\ \text{(by (3.36))} &= e^{-\beta t}\left[\int_0^t (\alpha - w_L(s))\,e^{\beta s}\,\mathrm{d}s + y_0\right] + \ln(\beta+r) + \\ &\quad -(\beta+r)\int_0^t \frac{B(\beta+r)e^{(\beta+r)s} + e^{\beta(s-t)}}{B(\beta+r)e^{(\beta+r)s}+1}\,\mathrm{d}s - \ln\left(B(\beta+r)+1\right) \\ &<^\dagger e^{-\beta t}\left[\int_0^t (\alpha - w_L(s))\,e^{\beta s}\,\mathrm{d}s + y_0\right] + \ln(\beta+r) - (\beta+r)\int_0^t e^{\beta(s-t)}\,\mathrm{d}s \\ &= y^0(t) + \ln w_F^0(t), \end{aligned} \tag{3.37}$$

where in the inequality "$<^\dagger$" we use $B(\beta + r) > 0$ and the fact that, for every h and k positive[3] we have,

$$-\frac{k+h}{k+1} \leq -h \quad \Leftrightarrow \quad h \leq 1.$$

Clearly relation (3.37) implies (3.35). By (3.34), we have

$$\begin{aligned} w_F^0(t) &= \beta + r \\ y^0(t) &= y_0 e^{-\beta t} + \frac{\alpha - (\beta+r)}{\beta}(1 - e^{-\beta t}) - \int_0^t w_L(s)e^{\beta(s-t)}\,\mathrm{d}s \\ \lambda_{Fc}(t) &= \frac{1}{\beta+r} \end{aligned} \tag{3.38}$$

for every w_L fixed. Since such w_L is generic, we are not in the position to guarantee that $y^0(t) \geq \ln 2$, i.e. $(w_L, w_F) \in \mathcal{A}_{OL}$. However, if $\mathcal{R}^F(w_L)$ is nonempty, then $\mathcal{R}^F(w_L) = \{w_F\}$ with w_F as in (3.38).

[3] in our case $h = e^{\beta(s-t)}$

Let us pass to **the point of view of the Leader–father**. For every strategy of the Leader–father, the Follower–son consider the strategy in w_F in (3.38). The current Hamiltonians for the Leader is similar to the definition in (3.8), taking into account that we use the current adjoint equation for the Follower:

$$H_L^c = y + \ln w_L + \lambda_{1Lc}(\alpha - w_L - (\beta + r) - \beta y) + \lambda_{2Lc}((\beta + r)\lambda_{Fc} - 1).$$

We have to guarantee the following conditions:

$$\nu_L(t) \in \arg\max_{v\geq 0} H_L^c = \arg\max_{v\geq 0}(\ln v - v\lambda_{1Lc}(t)) = \begin{cases} \frac{1}{\lambda_{1Lc(t)}} & \text{if } \lambda_{1Lc}(t) > 0 \\ \nexists & \text{if } \lambda_{1Lc(t)} \leq 0 \end{cases} \tag{3.39}$$

$$\dot{\lambda}_{1Lc} = r\lambda_{1Lc} - \frac{\partial H_L^c}{\partial y} = (r + \beta)\lambda_{1Lc} - 1 \tag{3.40}$$

$$\dot{\lambda}_{2Lc} = r\lambda_{2Lc} - \frac{\partial H_L^c}{\partial \lambda_{Fc}} = -\beta\lambda_{2Lc} \tag{3.41}$$

$$\lambda_{2Lc}(0) = 0 \tag{3.42}$$

Note that (3.41) and (3.42) imply $\lambda_{2Lc}(t) = 0$. Moreover the two conditions (3.39)–(3.40) are exactly the same of the two conditions (3.29)–(3.30): all the same arguments of before used to obtain the strategy for the follower in (3.38) can be used to show that the candidate to be the optimal strategy for the Leader, the associated trajectory and multiplier are

$$w_L(t) = \beta + r \tag{3.43}$$

$$y(t) = y_0 e^{-\beta t} + \frac{\alpha - 2(\beta + r)}{\beta}(1 - e^{-\beta t}) \tag{3.44}$$

$$\lambda_{1Lc}(t) = \frac{1}{\beta + r}$$

$$\lambda_{2Lc}(t) = 0$$

Let us check that $y(t) \geq \ln 2$: in fact, by plotting the function y and by (3.28), we have

$$y(t) \geq \min\left(y_0, \frac{\alpha - 2(\beta + r)}{\beta}\right) \geq 1, \qquad t \geq 0.$$

Now, considering

$$w_L^*(t) = w_F^*(t) = \beta + r$$

$$y^*(t) = y_0 e^{-\beta t} + \frac{\alpha - 2(\beta + r)}{\beta}(1 - e^{-\beta t})$$

$$\lambda_{Fc}^*(t) = \lambda_{1Lc}^*(t) = \frac{1}{\beta + r}$$

$$\lambda_{2Lc}^*(t) = 0$$

we know that $(w_L^*, w_F^*) \in \mathcal{A}_{OL}$; in order to conclude our problem, we have to prove that $\mathcal{R}^F(w_L^*) = \{w_F^*\}$ and that w_L is optimal for the point of view of the Leader. In order to do that, note that

$$\lim_{t\to\infty} \lambda_F^*(t)(y(t) - y^*(t)) \geq \lim_{t\to\infty} \frac{1}{\beta + r} e^{-rt}(\ln 2 - y^*(t)) = 0$$

$$\lim_{t\to\infty} \lambda_{1L}^*(t)(y(t) - y^*(t)) \geq \lim_{t\to\infty} \frac{1}{\beta + r} e^{-rt}(\ln 2 - y^*(t)) = 0$$

$$\lim_{t\to\infty} \lambda_{2L}^*(t)(\lambda_{Fc}(t) - \lambda_{Fc}^*(t)) = \lim_{t\to\infty} 0(\lambda_{Fc}(t) - \lambda_{Fc}^*(t)) = 0.$$

Finally it is easy to see that the Hamiltonians

$$(y, w_F) \mapsto H_1^c(y, w_F, w_L^*(t), \lambda_{Fc}^*(t)) \quad \text{and} \quad (y, \lambda_{Fc}, w_L) \mapsto H_2^c(y, \lambda_{Fc}, w_F(t), w_L, \lambda_{1Lc}^*(t), \lambda_{2Lc}^*(t))$$

are concave functions, for every fixed t, and that the control sets $U_1 = U_2 = [0, \infty)$ are convex. Hence (w_L^*, w_F^*) is a Stackelberg equilibrium.

Chapter 4

Two-persons zero-sum games

We are interested in the game

$$\begin{cases} \text{Player I: } \max_{\mathbf{u}_1} J(\mathbf{u}_1, \mathbf{u}_2), \qquad \text{Player II: } \min_{\mathbf{u}_2} J(\mathbf{u}_1, \mathbf{u}_2) \\ J(\mathbf{u}_1, \mathbf{u}_2) = \int_0^T f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}t + \psi(\mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \end{cases} \tag{4.1}$$

where T is fixed and U_1 and U_2 are the two closed control sets for the players.
Note that Player I, whose control is $\mathbf{u}_1$, wants to maximize the functional J; Player II has the control $\mathbf{u}_2$ and wants to minimize J. This is a two-persons zero-sum differential game. In this context

Definition 4.1. *A pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{OL}$, with $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}_0)$, is a* ***Nash equilibrium within the class of open–loop strategies*** *$\mathcal{A}_{OL}$ for (4.1) if*

$$J(\mathbf{u}_1, \mathbf{u}_2^*) \le J(\mathbf{u}_1^*, \mathbf{u}_2^*) \le J(\mathbf{u}_1^*, \mathbf{u}_2) \tag{4.2}$$

for every $(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{OL}$ and for every $(\mathbf{u}_1^*, \mathbf{u}_2) \in \mathcal{A}_{OL}$.
A pair of control functions $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$, is a ***Nash equilibrium within the class of feedback strategies*** *$\mathcal{A}_{FB}$ for (4.1) if (4.2) holds for every $(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$ and for every $(\mathbf{u}_1^*, \mathbf{u}_2) \in \mathcal{A}_{FB}$.*

Relation (4.2) implies that $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a saddle-point for J.

4.1 Open–loop Nash equilibria with the variational approach

In this case, the variational approach is useful and the Pontryagin necessary condition is as follows (see Theorem 6.13 in [2]):

Theorem 4.1. *Let us consider the problem (4.1) with f, g and ψ in C^1. Let $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{OL}$ be a Nash equilibrium with $\mathbf{x}^*$ associated trajectory.*
Then there exists a continuous multiplier $\boldsymbol{\lambda}^* : [0, T] \to \mathbb{R}^n$ such that

i. (min–max principle) for all $t \in [0, T]$, $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$

$$\begin{aligned} H(t, \mathbf{x}^*(t), \mathbf{u}_1, \mathbf{u}_2^*(t), \boldsymbol{\lambda}^*(t)) \le \\ \le H(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2^*(t), \boldsymbol{\lambda}^*(t)) \le \\ \le H(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2, \boldsymbol{\lambda}^*(t)); \end{aligned}$$

ii. (adjoint equation) in $[0, T]$ we have $\dot{\boldsymbol{\lambda}}^* = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}_1^*, \mathbf{u}_2^*, \boldsymbol{\lambda}^*)$;

iii. (transversality condition) $\boldsymbol{\lambda}^*(T) = \nabla_{\mathbf{x}}\psi(\mathbf{x}^*(T))$,

where H *is the* Hamiltonian function H *defined by*[1]

$$H(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \boldsymbol{\lambda}) = f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2).$$

Now, in order to give sufficient conditions to obtain a Nash equilibrium in the class $\mathcal{A}_{OL}$, we apply Theorem A.2 and A.3 to the two players, taking into account that the first one maximizes, while the second minimizes.

In some situation, there exists a Nash equilibrium with the class of feedback strategies for the problem (4.1), while the Nash equilibrium in the class of the open–loop strategies does not exists (see for example "the lady in the lake" in subsection 4.4.1). In this situation the previous Theorem 4.1 and the variational approach is not useful for the reasons explained in subsection 2.2. Even if an open–loop Nash equilibrium does not exists, a version of the previous theorem can still be utilized to obtain the feedback Nash equilibrium: let us give the details since this approach is largely adopted in the literature for solving pursuit–evasion games. Let $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where

$$\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t)), \tag{4.3}$$

with the corresponding trajectory $\mathbf{x}^*$ with $\mathbf{x}(0) = \boldsymbol{\alpha}$, be a Nash equilibrum within the class of feedback strategies for the game (4.1). Moreover let us suppose that a Nash equilibrium in the class of open–loop strategies does not exists. However, let us consider the function

$$(\mathbf{u}_1^*(t), \mathbf{u}_2^*(t)) \tag{4.4}$$

given by (4.3); this function is not a open–loop strategies, since by definition a open–loop strategy is a function which depends only on t and $\mathbf{x}_0$. This function is usually called **open–loop representation of the feedback strategy**. We have (see Theorem 8.2 in [2])

Theorem 4.2. *As in Theorem 4.1, let us consider the problem (4.1) with* f, g *and* ψ *in* C^1. *We assume that there exists a Nash equilibrum within the class of feedback strategies* $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, *with* $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$, *with* $\mathbf{x}^*$ *associated trajectory. Let us consider its open–loop representation in (4.4). Then there exists a continuous multiplier* $\boldsymbol{\lambda}^* : [0, T] \to \mathbb{R}^n$ *such that* i.–iii. *in Theorem 4.1 are satisfied.*

This result will play a fundamental role in "the lady in the lake" in subsection 4.4.1.

4.1.1 War of attrition and attack

This model is due to Isaacs (see section 5.4 in [10] and page 91 in [6]). We assume that two opponents A and B are at war with each other, for a very long time. Let us define $x_1 = x_1(t)$ and $x_2 = x_2(t)$ the supply of resources for A and B respectively, at time t. Each player at each time can devote some fraction of the efforts, ($\alpha = \alpha(t)$ for Player A, $\beta = \beta(t)$ Player for B) to attrition (= guerrilla warfare, for example to destroy the production of resources of the competitor) and the remaining fraction ($1 - \alpha$ and $1 - \beta$ respectively) to direct attack. Clearly α and β have values in $[0, 1]$.

Let us introduce m_i the constant rate of production of war material for the two players, c_1 the effectiveness of B's weapons against A's production and c_2 the effectiveness of A's weapons against B's production. We will assume $c_2 > c_1$, a hypothesis that introduces an asymmetry into the problem. The dynamics are governed by the system of ODE

$$\begin{cases} \dot{x}_1 = m_1 - c_1\beta x_2 \\ \dot{x}_2 = m_2 - c_2\alpha x_1 \end{cases}$$

The A opponent want to realize an advantage with respect to B in the direct attack, i.e

$$\max \int_0^T [(1 - \alpha)x_1 - (1 - \beta)x_2] \, \mathrm{d}t;$$

[1]Note that in our problem (4.1), the final time T is fixed and the trajectory in such final time, i.e. $\mathbf{x}(T)$, is free: hence we can set $\lambda_0^* = 1$.

the B opponent want to realize an advantage with respect to A in the direct attack, i.e

$$\max \int_0^T [(1-\beta)x_2 - (1-\alpha)x_1]\, \mathrm{d}t = -\min \int_0^T [(1-\alpha)x_1 - (1-\beta)x_2]\, \mathrm{d}t.$$

Hence we have the following two-persons zero-sum game

$$\begin{cases} \textit{Player A: } \max_\alpha J(\alpha,\beta), & \textit{Player B: } \min_\beta J(\alpha,\beta) \\ 0 \le \alpha \le 1 & 0 \le \beta \le 1 \\ J(\alpha,\beta) = \int_0^T [(1-\alpha)x_1 - (1-\beta)x_2]\, \mathrm{d}t \\ \dot{x}_1 = m_1 - c_1\beta x_2 \\ \dot{x}_2 = m_2 - c_2\alpha x_1 \\ x_i(0) = x_{i0} > 0 \end{cases}$$

The final time T is very large and fixed. Note that it is reasonable to require that $x_i(t) > 0$ but, in order to simplify our solution, we remove it.

Let us looking for some Nash equilibrium in the family of open–loop strategies, using the variational approach. The Hamiltonian $H = H(t, x_1, x_2, \alpha, \beta, \lambda_1, \lambda_2)$ is

$$H = (1-\alpha)x_1 - (1-\beta)x_2 + \lambda_1(m_1 - c_1\beta x_2) + \lambda_2(m_2 - c_2\alpha x_1).$$

Note that the final value of the trajectory (x_1, x_2) is free and hence we put $\lambda_0 = 1$. We have to guarantee the conditions of Theorem 4.1:

$$\alpha \in \arg\max_{a\in[0,1]} H = \arg\max_{a\in[0,1]} a(-1 - c_2\lambda_2) = \begin{cases} 1 & \text{if } \lambda_2 < -\frac{1}{c_2} \\ ? & \text{if } \lambda_2 = -\frac{1}{c_2} \\ 0 & \text{if } \lambda_2 > -\frac{1}{c_2} \end{cases} \tag{4.5}$$

$$\beta \in \arg\min_{b\in[0,1]} H = \arg\min_{b\in[0,1]} b(1 - c_1\lambda_1) = \begin{cases} 0 & \text{if } \lambda_1 < \frac{1}{c_1} \\ ? & \text{if } \lambda_1 = \frac{1}{c_1} \\ 1 & \text{if } \lambda_1 > \frac{1}{c_1} \end{cases} \tag{4.6}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -(1-\alpha) + c_2\alpha\lambda_2 \tag{4.7}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = (1-\beta) + c_1\beta\lambda_1 \tag{4.8}$$

$$\lambda_1(T) = \lambda_2(T) = 0 \tag{4.9}$$

First, let us notice that in arg max and arg min in (4.5) and (4.6) we use the condition $x_i(t) > 0$. Moreover, we remark that the arg max and arg min in (4.5) and (4.6) do not depend on (x_1, x_2): hence we are in the position to looking for a Nash equilibrium in the family of open–loop strategies.

The adjoint equation (4.7) and the Maximum Principle (4.5) imply that λ_1 is a decrease function since $\dot{\lambda}_1 \le -1$: indeed

$$\begin{array}{lll} \text{if } \lambda_2 < -1/c_2 & \Rightarrow\ \alpha = 1 & \Rightarrow\ \dot{\lambda}_1 = -(1-\alpha) + c_2\alpha\lambda_2 < -1 \\ \text{if } \lambda_2 = -1/c_2 & \Rightarrow\ \dot{\lambda}_1 = -(1-\alpha) + c_2\alpha\lambda_2 = -1 & \\ \text{if } \lambda_2 > -1/c_2 & \Rightarrow\ \alpha = 0 & \Rightarrow\ \dot{\lambda}_1 = -(1-\alpha) + c_2\alpha\lambda_2 = -1 \end{array}$$

A similar argument, using the adjoint equation (4.8) and the Minimum Principle (4.6), implies that λ_2 is an increase function since $\dot{\lambda}_2 \ge 1$.

Now, by (4.9) there exists $\tau \in [0, T)$ such that

$$\lambda_1(t) < \frac{1}{c_1}, \qquad \lambda_2(t) > -\frac{1}{c_2}, \qquad \forall t \in (\tau, T], \tag{4.10}$$

i.e. $\alpha = \beta = 0$ in $(\tau, T]$. The adjoint equation $\dot\lambda_1 = -1$, $\dot\lambda_2 = 1$, and the transversality conditions (4.9) give

$$\lambda_1(t) = T - t, \qquad \lambda_2(t) = t - T;$$

the assumptions in (4.10), taking into account that $c_2 > c_1$, imply

$$\tau = T - \frac{1}{c_2} \qquad \text{in } (\tau, T]. \tag{4.11}$$

Now, let us suppose that there exists $\tau' \in [0, \tau)$ such that

$$\lambda_1(t) < \frac{1}{c_1}, \qquad \lambda_2(t) < -\frac{1}{c_2}, \qquad \forall t \in (\tau', \tau) \tag{4.12}$$

i.e. $\alpha = 1$, $\beta = 0$ in (τ', τ). The adjoint equation $\dot\lambda_1 = c_2\lambda_2$, $\dot\lambda_2 = 1$, and the continuity of the multipliers in the point $t = \tau$ give

$$\lambda_1(t) = \frac{c_2}{2}(T-t)^2 + \frac{1}{2c_2}, \qquad \lambda_2(t) = t - T; \qquad \text{in } (\tau', \tau].$$

the assumptions in (4.12), taking into account that $2c_2 > c_2 > c_1$, imply

$$\tau' = T - \frac{1}{c_2}\sqrt{\frac{2c_2 - c_1}{c_1}}. \tag{4.13}$$

Now, let us suppose that there exists $\tau'' \in [0, \tau')$ such that

$$\lambda_1(t) > \frac{1}{c_1}, \qquad \lambda_2(t) < -\frac{1}{c_2}, \qquad \forall t \in (\tau'', \tau') \tag{4.14}$$

i.e. $\alpha = \beta = 1$ in (τ'', τ'). The adjoint equation give

$$\dot\lambda_1 = c_2\lambda_2 \qquad \text{and} \qquad \dot\lambda_2 = c_1\lambda_1 \tag{4.15}$$

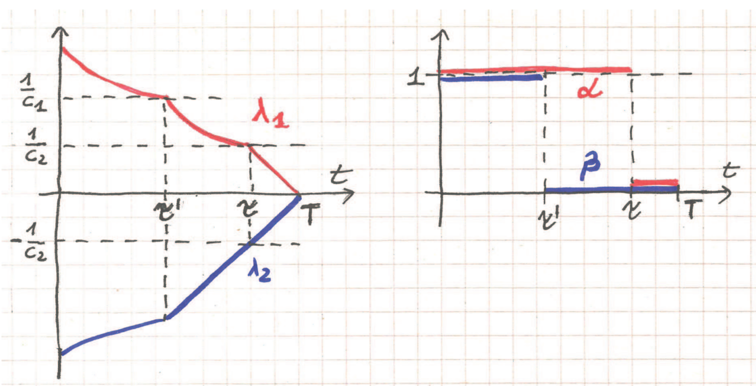
and hence

$$\begin{aligned} \ddot\lambda_1 - c_1c_2\lambda_1 = 0 \qquad &\Rightarrow \quad \lambda_1 = Ae^{\sqrt{c_1c_2}t} + Be^{-\sqrt{c_1c_2}t} \\ &\Rightarrow \quad \lambda_2 = \sqrt{\frac{c_1}{c_2}}\left(Ae^{\sqrt{c_1c_2}t} - Be^{-\sqrt{c_1c_2}t}\right) \end{aligned}$$

for some constants A and B. Using the continuity of the multipliers in the point $t = \tau'$ we have, taking into account (4.13),

$$\begin{aligned} \lambda_1(t) &= \left(1 - \sqrt{\frac{2c_2 - c_1}{c_2}}\right)\frac{e^{\sqrt{c_1c_2}(t-\tau')}}{2c_1} + \left(1 + \sqrt{\frac{2c_2 - c_1}{c_2}}\right)\frac{e^{-\sqrt{c_1c_2}(t-\tau')}}{2c_1} \\ \lambda_2(t) &= \left(1 - \sqrt{\frac{2c_2 - c_1}{c_2}}\right)\frac{e^{\sqrt{c_1c_2}(t-\tau')}}{2\sqrt{c_1c_2}} - \left(1 + \sqrt{\frac{2c_2 - c_1}{c_2}}\right)\frac{e^{-\sqrt{c_1c_2}(t-\tau')}}{2\sqrt{c_1c_2}} \end{aligned}$$

Since λ_1 is a decreasing function and λ_2 is an increasing function, we obtain $\tau'' = 0$.



The pair (α^*, β^*) candidate to be a Nash equilibrium is

$$\alpha^*(t) = \begin{cases} 1 & \text{if } t \in [0, \tau] \\ 0 & \text{if } t \in (\tau, T] \end{cases} \qquad \beta^*(t) = \begin{cases} 1 & \text{if } t \in [0, \tau'] \\ 0 & \text{if } t \in (\tau', T] \end{cases}$$

where τ and τ' is defined in (4.13) and (4.11). Since (α^*, β^*) is constant, except two points, and consequently the dynamics is linear in (x_1, x_2) with constant coefficients, except such two points, then there exists a unique solution (x_1^*, x_2^*) of such ODE with initial data $x_i(0) = x_{i0}$ and the strategy (α^*, β^*) is admissible. Let $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*)$ be the multiplier obtained by the previous calculations.

Let us note that the function $(t, x_1, x_2, \alpha, \beta) \mapsto H(t, x_1, x_2, \alpha, \beta, \lambda_1^*(t), \lambda_2^*(t))$, for a fixed t, is not concave in (x_1, x_2, α) variable and convex in (x_1, x_2, β) variable: let us use Theorem A.3 in order to guarantee some sufficient condition of optimality. First we consider the maximed Hamiltonian for the Player A:

$$\begin{aligned} H_A^0(t, x_1, x_2, \lambda_1^*(t), \lambda_2^*(t)) &= \max_{a \in [0,1]} H(t, x_1, x_2, a, \beta^*(t), \lambda_1^*(t), \lambda_2^*(t)) \\ &= x_2 \beta^*(t)(1 - c_1 \lambda_1^*(t)) + x_1 - x_2 + m_1 \lambda_1^*(t) + m_2 \lambda_2^*(t) + x_1 [\max_{a \in [0,1]} a(-1 - c_2 \lambda_2^*(t))] \end{aligned}$$

It is easy to see that, for every fixed t, the function $(x_1, x_2) \mapsto H_A^0(t, x_1, x_2, \lambda_1^*(t), \lambda_2^*(t))$ is linear and hence concave in (x_1, x_2): hence α^* is really a optimal solution for the max problem of the First Player, with β^* fixed. A similar argument holds for the minimized Hamiltonian

$$H_B^0(t, x_1, x_2, \lambda_1^*(t), \lambda_2^*(t)) = \min_{b \in [0,1]} H(t, x_1, x_2, \alpha^*(t), b, \lambda_1^*(t), \lambda_2^*(t)),$$

showing that, for every fixed t, the function $(x_1, x_2) \mapsto H_B^0(t, x_1, x_2, \lambda_1^*(t), \lambda_2^*(t))$ is linear and hence convex in (x_1, x_2). Hence β^* is a optimal solution for the min problem of the Player B, with α^* fixed. We obtain that (α^*, β^*) is a Nash equilibrium in the class of open-loop strategies.

4.2 An introduction to upper and lower value functions with the DP

We intend now to study Nash equilibria in the class of feedback strategies; in this context we know that the variational approach is not useful (see subsection 2.2). The idea is to use the Dynamic Programming (DP) and to study the value functions. However, the general theory about the value functions with respect to the feedback strategies is very complicated and long: since our aim is to give only an idea of the problems and the tools, **in this section we concentrate our attention only on open-loop strategies with Dynamic Programming approach**. The definitions and the proofs of the main results are mostly based on the Belmann-Hamilton-Jacobi equations and such results can be generalized in the class $\mathcal{A}_{FB}$ (see for example chapter 8 in [1]). In the next sections we will consider the class $\mathcal{A}_{FB}$, using the idea and generalizations of this section.

4.2.1 Upper and lower value functions for open-loop strategies

Let us consider the problem (4.1) and the following two assumptions:

1. $f : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ are bounded and uniformly continuous with

$$\begin{aligned} &|f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)| \le C_1 \qquad |f(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) - f(t, \mathbf{x}', \mathbf{u}_1, \mathbf{u}_2)| \le C_1 \|\mathbf{x} - \mathbf{x}'\|, \\ &\|g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\| \le C_1 \qquad \|g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) - g(t, \mathbf{x}', \mathbf{u}_1, \mathbf{u}_2)\| \le C_1 \|\mathbf{x} - \mathbf{x}'\|, \\ &|\psi(\mathbf{x})| \le C, \qquad |\psi(\mathbf{x}) - \psi(\mathbf{x}')| \le C \|\mathbf{x} - \mathbf{x}'\|, \end{aligned}$$

   for some constant C and for every $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, $\mathbf{u}_1 \in U_1$, $\mathbf{u}_2 \in U_2$;

2. the control sets U_i are compacts; more precisely we assume $U_i \subset B_{\mathbb{R}^{k_i}}(0, R_i)$ for some fixed and positive R_i.

For the Player I, let us introduce the **set of controls at time** τ, with $\tau \in [0,T]$ fixed, as

$$\mathcal{U}_1(\tau) = \{\mathbf{u}_1 : [\tau, T] \to U_1, \text{ measurable}\}.$$

In a similar way, we define $\mathcal{U}_2(\tau) = \{\mathbf{u}_2 : [\tau, T] \to U_2, \text{ measurable}\}$.

Remark 4.1. *By the previous assumptions 1. on g and 2., for every* $(\tau, \boldsymbol{\xi})$ *and* $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{U}_1(\tau) \times \mathcal{U}_2(\tau)$, *then* $(\mathbf{u}_1, \mathbf{u}_2)$ *is admissible, i.e. there exists a unique solution* $\mathbf{x}$ *of*

$$\begin{cases} \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) & \textit{for a.e. } t \in [\tau, T] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

Let us introduce our a new notion of strategies that is useful in the zero-sum games, in the particular case of decision rules ν_i for the two players such that they depend only on the time t, i.e. $\boldsymbol{\nu}_i = \boldsymbol{\nu}_i(t)$:

Definition 4.2. *Let us fix* $\tau \in [0,T]$. *A map*

$$\Phi_1 : \mathcal{U}_2(\tau) \to \mathcal{U}_1(\tau)$$

is a ***nonanticipative strategy for the Player I at time*** τ *if, for any time* $t \in [\tau, T]$ *and any controls* $\mathbf{u}_2, \mathbf{u}_2' \in \mathcal{U}_2(\tau)$ *such that* $\mathbf{u}_2 = \mathbf{u}_2'$ *almost everywhere in* $[\tau, t]$, *then we have* $\Phi_1[\mathbf{u}_2] = \Phi_1[\mathbf{u}_2']$ *almost everywhere in* $[\tau, t]$. *We denote by* $\mathcal{S}_1(\tau)$ ***the set of such nonanticipative strategies at time*** τ ***for the Player I.***

In a symmetric way we denote by $\mathcal{S}_2(\tau)$ *the set of Player II nonanticipative strategies, which are the nonanticipative maps*

$$\Phi_2 : \mathcal{U}_1(\tau) \to \mathcal{U}_2(\tau)$$

Note that for every $(\tau, \boldsymbol{\xi})$, $\mathbf{u}_1 \in \mathcal{U}_1(\tau)$ and $\Phi_2 \in \mathcal{S}_2(\tau)$ we have, by remark 4.1 that $(\mathbf{u}_1, \Phi_2[\mathbf{u}_1])$ is admissible; similar property holds for $(\Phi_1[\mathbf{u}_2], \mathbf{u}_2)$.
Moreover, it is clear that for every $\mathbf{u}_1 \in \mathcal{U}_1(0)$ and $\Phi_2 \in \mathcal{S}_2(0)$ we have that $(\mathbf{u}_1, \Phi_1[\mathbf{u}_1])$ is a open–loop strategy in the class $\mathcal{A}_{OL}$ as in Definition 1.5 for the problem (4.1); similar property holds for $(\Phi_1[\mathbf{u}_2], \mathbf{u}_2) \in \mathcal{A}_{OL}$.
The simplest example of nonanticipative strategy Φ_1 for the Player I at time τ is the constant one: more precisely, let us fix $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(\tau)$ and let us define the constant strategy $\Phi_1 = \Phi_1^{\widetilde{\mathbf{u}}_1}$ by

$$\Phi_1[\mathbf{u}_2] = \widetilde{\mathbf{u}}_1, \qquad \forall \mathbf{u}_2 \in \mathcal{U}_2(\tau)$$

For the problem (4.1), in the assumptions *1.* and *2.*, let us define the two value functions as follows (see for example Definition 1.6 of Chapter VIII in [1]):

Definition 4.3. *Let us consider the problem (4.1) with the assumptions 1. and 2. The* ***lower value function*** $V^- : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is defined by*

$$V^-(\tau, \boldsymbol{\xi}) = \inf_{\Phi_2 \in \mathcal{S}_2(\tau)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(\tau)} \int_\tau^T f(t, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1])\, \mathrm{d}t + \psi(\mathbf{x}(T)),$$

where $\mathbf{x}$ *is the trajectory associated to the control* $(\mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \in \mathcal{U}_1(\tau) \times \mathcal{U}_2(\tau)$ *with initial data* $\mathbf{x}(\tau) = \boldsymbol{\xi}$.

The ***upper value function*** $V^+ : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is defined as*

$$V^+(\tau, \boldsymbol{\xi}) = \sup_{\Phi_1 \in \mathcal{S}_1(\tau)} \inf_{\mathbf{u}_2 \in \mathcal{U}_2(\tau)} \int_\tau^T f(t, \mathbf{x}, \Phi_1[\mathbf{u}_2], \mathbf{u}_2)\, \mathrm{d}t + \psi(\mathbf{x}(T)).$$

where $\mathbf{x}$ *is the trajectory associated to the control* $(\Phi_1[\mathbf{u}_2], \mathbf{u}_2) \in \mathcal{U}_1(\tau) \times \mathcal{U}_2(\tau)$ *with initial data* $\mathbf{x}(\tau) = \boldsymbol{\xi}$.

Note that assumptions *1.* on g and *2.* imply that every pair $(\Phi_1[\mathbf{u}_2], \mathbf{u}_2)$ and $(\mathbf{u}_1, \Phi_2[\mathbf{u}_1])$ is admissible and hence we are considering sup and inf on a nonempty sets; moreover, assumption *1.* on f and ψ implies that V^+ and V^- are bounded.

One of the two players announces his strategy in response to the other's choice of control, the other player chooses the control. The player who "plays second", i.e., who chooses the strategy, has an advantage. In the definition of V^+, the Player II choices its nonanticipative strategies Φ_2 and "after" the Player I choices its optimal $\mathbf{u}_1$. Hence we have that[2]

Remark 4.2. *For every* $(\tau, \boldsymbol{\xi})$ *we have*

$$V^-(\tau, \boldsymbol{\xi}) \le V^+(\tau, \boldsymbol{\xi}) \tag{4.16}$$

In general V^+ and V^- are different functions (as we will see in Example 4.2.1). The next example is in [1]:

Example 4.2.1. *Let us consider the problem*

$$\begin{cases} \text{Player I: } \max_{u_1} J(u_1, u_2), \qquad \text{Player II: } \min_{u_2} J(u_1, u_2) \\ |u_1| \le 1 \qquad\qquad\qquad\qquad |u_2| \le 1 \\ J(u_1, u_2) = \displaystyle\int_0^\infty \operatorname{sgn}(x)\left(1 - e^{-|x|}\right) e^{-t}\, \mathrm{d}t \\ \dot{x} = (u_1 - u_2)^2 \\ x(0) = x_0 \end{cases}$$

By definition we have

$$\begin{aligned} V^-(0, \xi) &= \inf_{\Phi_2 \in \mathcal{S}_2(0)} \sup_{u_1 \in \mathcal{U}_1(0)} \int_0^\infty f(t, x)\, \mathrm{d}t \\ V^+(0, \xi) &= \sup_{\Phi_1 \in \mathcal{S}_1(0)} \inf_{u_2 \in \mathcal{U}_2(0)} \int_0^\infty f(t, x)\, \mathrm{d}t \end{aligned}$$

where $f(t, x) = \operatorname{sgn}(x)\left(1 - e^{-|x|}\right) e^{-t}$ *and* x *is the trajectory associated with* $x(0) = \xi$*. We show that, for every* $\xi > 0$*, we have*

$$V^-(0, \xi) < V^+(0, \xi) \tag{4.17}$$

Let us fix $\xi \ge 0$*. First we note that the dynamics gives* $x(t) \ge \xi$*, for every* (u_1, u_2)*. Moreover, it is easy to see that the function* $x \mapsto f(t, x)$ *is increasing, for every fixed* t*.*

For every $u_1 \in \mathcal{U}_1(0)$*, where* $\mathcal{U}_i(0) = \{u : [0, \infty) \to [-1, 1], \text{ measurable}\}$*, let us consider the nonanticipative strategy* $\widetilde{\Phi}_2 : \mathcal{U}_1(0) \to \mathcal{U}_2(0)$ *defined by* $\widetilde{\Phi}_2[u_1] = u_1$*. Hence the trajectory* x *associated to such pair* $(u_1, \widetilde{\Phi}_2[u_1])$ *is the solution of*

$$\begin{cases} \dot{x} = (u_1 - \widetilde{\Phi}_2[u_1])^2 = 0 \\ x(0) = \xi \end{cases}$$

hence $x(t) = \xi$*. Clearly, since* $x \mapsto f(t, x)$ *is increasing and* $x(t) \ge \xi$*, such particular strategy* $\widetilde{\Phi}_2$ *for the second Player is the best possible (remember that the second Player want to minimize). Hence we obtain*

$$\begin{aligned} V^-(0, \xi) &\le \sup_{u_1 \in \mathcal{U}_1(0)} \int_0^\infty f(t, x)\, \mathrm{d}t \qquad (\text{we choose } \Phi_2 = \widetilde{\Phi}_2) \\ &= \int_0^\infty \left(1 - e^{-\xi}\right) e^{-t}\, \mathrm{d}t \\ &= 1 - e^{-\xi} \end{aligned} \tag{4.18}$$

Now for every $u_2 \in \mathcal{U}_2(0)$ *let us consider the nonanticipative strategy* $\widetilde{\Phi}_1 : \mathcal{U}_2(0) \to \mathcal{U}_1(0)$ *defined by*

$$\widetilde{\Phi}_1[u_2](t) = \begin{cases} +1 & \text{if } u_2(t) \le 0 \\ -1 & \text{if } u_2(t) > 0 \end{cases}$$

Hence the trajectory x *associated to such pair* $(\widetilde{\Phi}_1[u_2], u_1)$ *is the solution of*

$$\begin{cases} \dot{x} = (\widetilde{\Phi}_1[u_2] - u_2)^2 = (1 + |u_2|)^2 \\ x(0) = \xi \end{cases}$$

[2] The rigorous proof of this remark follows by further considerations.

Clearly, for such particular strategy $\widetilde{\Phi}_1$ for the first Player, the optimal control for the second player is $u_2 = 0$; in this case the associated trajectory is $x(t) = \xi + t$. Hence we obtain

$$\begin{aligned} V^+(0,\xi) &\geq \inf_{u_2\in\mathcal{U}_2(0)} \int_0^\infty f(t,x)\,\mathrm{d}t \qquad (\text{we choose } \Phi_1 = \widetilde{\Phi}_1) \\ &= \int_0^\infty \left(1 - e^{-(\xi+t)}\right) e^{-t}\,\mathrm{d}t \qquad (\text{we choose } u_2 = 0) \\ &= 1 - \frac{e^{-\xi}}{2} \end{aligned} \tag{4.19}$$

Hence, by (4.18) and (4.19), we obtain

$$V^+(0,\xi) \geq 1 - \frac{e^{-\xi}}{2} > 1 - e^{-\xi} \geq V^-(0,\xi), \qquad \forall \xi \geq 0$$

i.e. relation (4.17).

Definition 4.4. *We say that the game (4.1) has value function $V : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ when (4.16) is an equality, for every $(\tau, \boldsymbol{\xi})$, and in this case we set*

$$V(t,\mathbf{x}) = V^+(t,\mathbf{x}) = V^-(t,\mathbf{x}),$$

*for every $(t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n$. In this case, V is the **value function** for the problem (4.1).*

Clearly, the problem to guarantees that (4.1) has value function is very interesting and crucial. We will discuss this problem in subsection 4.2.3

4.2.2 Isaacs' condition

In the next subsections we are interested on giving the main ideas of Isaacs theory. Let us start:

Definition 4.5. *Let us define the **upper Hamiltonian of Dynamic Programming** $H^+_{DP} : [0,T]\times\mathbb{R}^{2n} \to \mathbb{R}$ defined by*

$$H^+_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) = \min_{\mathbf{v}_2\in U_2}\max_{\mathbf{v}_1\in U_1}\Big(f(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2)\Big)$$

*and the **lower Hamiltonian of Dynamic Programming** $H^-_{DP} : [0,T]\times\mathbb{R}^{2n} \to \mathbb{R}$ defined by*

$$H^-_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) = \max_{\mathbf{v}_1\in U_1}\min_{\mathbf{v}_2\in U_2}\Big(f(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2)\Big).$$

We note that in the definition of V^+ we have a "sup–inf", while in the definition of H^+_{DP} we have a "min–max".

Remark 4.3. *We have*

$$H^-_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) \leq H^+_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) \tag{4.20}$$

Proof. Let us fix $(t,\mathbf{x},\boldsymbol{\lambda})$ and let us denote by h the function $h(\mathbf{v}_1,\mathbf{v}_2) = f(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) + \boldsymbol{\lambda}\cdot g(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2)$. Clearly

$$\min_{\mathbf{v}_2\in U_2} h(\mathbf{v}_1,\mathbf{v}_2) \leq h(\mathbf{v}_1,\mathbf{v}_2), \qquad \forall(\mathbf{v}_1,\mathbf{v}_2)\in U_1\times U_2.$$

This implies

$$\max_{\mathbf{v}_1\in U_1}\min_{\mathbf{v}_2\in U_2} h(\mathbf{v}_1,\mathbf{v}_2) \leq \max_{\mathbf{v}_1\in U_1} h(\mathbf{v}_1,\mathbf{v}_2), \qquad \forall \mathbf{v}_2\in U_2$$

and hence the thesis. □

The next example shows that inequality (4.20) can be strict.

Example 4.2.2. *Let us consider the problem in Example 4.2.1. Clearly we have*

$$H^+_{DP}(t,x,\lambda) = \min_{|v_2|\le 1}\max_{|v_1|\le 1}\left(\mathrm{sgn}(x)\left(1-e^{-|x|}\right)e^{-t} + \lambda(v_1-v_2)^2\right) = \mathrm{sgn}(x)\left(1-e^{-|x|}\right)e^{-t} + \min_{|v_2|\le 1}\max_{|v_1|\le 1}\lambda(v_1-v_2)^2$$

$$H^-_{DP}(t,x,\lambda) = \max_{|v_1|\le 1}\min_{|v_2|\le 1}\left(\mathrm{sgn}(x)\left(1-e^{-|x|}\right)e^{-t} + \lambda(v_1-v_2)^2\right) = \mathrm{sgn}(x)\left(1-e^{-|x|}\right)e^{-t} + \max_{|v_1|\le 1}\min_{|v_2|\le 1}\lambda(v_1-v_2)^2$$

First, let us fix λ *and* $v_2 \in [1,-1]$*: we have*

$$\max_{|v_1|\le 1}\lambda(v_1-v_2)^2 = \begin{cases} 0 & \text{if } \lambda \le 0 \\ \lambda(1+|v_2|)^2 & \text{if } \lambda > 0 \end{cases}$$

and hence

$$\min_{|v_2|\le 1}\max_{|v_1|\le 1}\lambda(v_1-v_2)^2 = \begin{cases} 0 & \text{if } \lambda \le 0 \\ \lambda & \text{if } \lambda > 0 \end{cases} \tag{4.21}$$

Now, let us fix λ *and* $v_1 \in [1,-1]$*: we have*

$$\min_{|v_2|\le 1}\lambda(v_1-v_2)^2 = \begin{cases} \lambda(1+|v_2|)^2 & \text{if } \lambda \le 0 \\ 0 & \text{if } \lambda > 0 \end{cases}$$

and hence

$$\max_{|v_1|\le 1}\min_{|v_2|\le 1}\lambda(v_1-v_2)^2 = \begin{cases} \lambda & \text{if } \lambda \le 0 \\ 0 & \text{if } \lambda > 0 \end{cases} \tag{4.22}$$

Inequalities (4.21) and (4.22) give that

$$H^-_{DP}(t,x,\lambda) < H^+_{DP}(t,x,\lambda), \qquad \forall \lambda \neq 0$$

The previous example suggests the following definition:

Definition 4.6. *We say that the minimax condition, or **Isaacs' condition**, is satisfied if*

$$H^-_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) = H^+_{DP}(t,\mathbf{x},\boldsymbol{\lambda})$$

for every $(t,\mathbf{x},\boldsymbol{\lambda})$*. In this case we define by* $H_{DP} : [0,T]\times\mathbb{R}^{2n} \to \mathbb{R}$ *the **Hamiltonian of Dynamic Programming** by*

$$H_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) = H^-_{DP}(t,\mathbf{x},\boldsymbol{\lambda}) = H^+_{DP}(t,\mathbf{x},\boldsymbol{\lambda}). \tag{4.23}$$

4.2.3 Upper and lower Isaacs' equations

At this point we are interested to study the two functions of V^+ and V^-. Let us start with their regularity.

Theorem 4.3. *Let us consider the problem (4.1) with the assumptions 1. and 2. Then* V^- *is bounded and uniformly Lipschitz continuous, i.e.*

$$|V^-(\tau,\boldsymbol{\xi}) - V^-(\tau',\boldsymbol{\xi}')| \le \widehat{C}\left(|\tau-\tau'| + \|\boldsymbol{\xi}-\boldsymbol{\xi}'\|\right),$$

for every $\tau,\tau' \in [0,T]$ *and* $\boldsymbol{\xi},\boldsymbol{\xi}' \in \mathbb{R}^n$*, for some constant* $\widehat{C}$*.*
A similar results holds for V^+*.*

Proof. (see Theorem 3.2 in [8]). Let us fix $\tau < \tau'$ in $[0,T]$ and $\boldsymbol{\xi},\boldsymbol{\xi}' \in \mathbb{R}^n$. It is immediate to see that

$$|V^-(\tau,\boldsymbol{\xi})| \le C_1 T + C_2.$$

Now let us fix $\epsilon > 0$. There exists $\widehat{\Phi}_2 \in \mathcal{S}_2(\tau)$ such that

$$V^-(\tau,\boldsymbol{\xi}) \ge \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)}\left\{\int_\tau^T f(t,\mathbf{x},\mathbf{u}_1,\widehat{\Phi}_2[\mathbf{u}_1])\,dt + \psi(\mathbf{x}(T))\right\} - \epsilon. \tag{4.24}$$

Fix $\mathbf{u}_{1fix} \in U_1$. For every $\mathbf{u}_1 \in \mathcal{U}_1(\tau')$, let us define $\widehat{\mathbf{u}} \in \mathcal{U}_1(\tau)$ by

$$\widehat{\mathbf{u}}(t) = \begin{cases} \mathbf{u}_{1fix}, & \text{for } t \in [\tau,\tau') \\ \mathbf{u}_1(t), & \text{for } t \in [\tau',T] \end{cases} \tag{4.25}$$

Let us define $\widetilde{\Phi}_2 \in \mathcal{S}_2(\tau')$ such that

$$\widetilde{\Phi}_2[\mathbf{u}_1](t) = \widehat{\Phi}_2[\widehat{\mathbf{u}}_1](t), \qquad \forall \mathbf{u}_1 \in \mathcal{U}_1(\tau'),\ t \in [\tau',T]$$

Clearly

$$V^-(\tau',\boldsymbol{\xi}') \le \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau')} \left\{ \int_{\tau'}^T f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\}.$$

Now there exists $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(\tau')$ such that

$$V^-(\tau',\boldsymbol{\xi}') \le \int_{\tau'}^T f(t,\mathbf{x},\widetilde{\mathbf{u}}_1,\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) + \epsilon. \tag{4.26}$$

And (4.24) gives

$$V^-(\tau,\boldsymbol{\xi}) \ge \int_{\tau}^T f(t,\mathbf{x},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) - \epsilon, \tag{4.27}$$

where $\widehat{\mathbf{u}}_1$ is defined by $\widetilde{\mathbf{u}}_1$ via relation (4.25). Note that the trajectories $\mathbf{x}$ that appear in (4.26) and in (4.27) are different function; in particular, denoting by $\widetilde{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ such trajectories in (4.26) and in (4.27) respectively, they solve

$$\begin{cases} \dot{\widetilde{\mathbf{x}}}(t) = g(t,\widetilde{\mathbf{x}}(t),\widetilde{\mathbf{u}}_1(t),\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](t)) & \text{a.e. in } [\tau',T] \\ \widetilde{\mathbf{x}}(\tau') = \boldsymbol{\xi}' \end{cases}$$

and

$$\begin{cases} \dot{\widehat{\mathbf{x}}}(t) = g(t,\widehat{\mathbf{x}}(t),\widehat{\mathbf{u}}_1(t),\widehat{\Phi}_2[\widehat{\mathbf{u}}_1](t)) & \text{a.e. in } [\tau,T] \\ \widehat{\mathbf{x}}(\tau) = \boldsymbol{\xi} \end{cases}$$

Clear we have, by the bounded assumption *2.* on g,

$$\|\boldsymbol{\xi} - \widehat{\mathbf{x}}(\tau')\| = \left\| \int_{\tau}^{\tau'} g(t,\widehat{\mathbf{x}},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t \right\| \le C_1(\tau'-\tau). \tag{4.28}$$

The Lipschitz assumption *2.* on g and since $\widetilde{\mathbf{u}}_1 = \widehat{\mathbf{u}}_1$ and $\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1] = \widehat{\Phi}_2[\widehat{\mathbf{u}}_1]$ on $[\tau',T]$, we have that, for every $t\in[\tau',T]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\mathbf{x}}(t)-\widetilde{\mathbf{x}}(t)\| &= \frac{\left(\widehat{\mathbf{x}}(t)-\widetilde{\mathbf{x}}(t), \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{x}}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t)\right)}{\|\widehat{\mathbf{x}}(t)-\widetilde{\mathbf{x}}(t)\|} \\
&\le \left\| \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{x}}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{x}}(t) \right\| \\
&= \left\| g(t,\widehat{\mathbf{x}}(t),\widehat{\mathbf{u}}_1(t),\widehat{\Phi}_2[\widehat{\mathbf{u}}_1](t)) - g(t,\widetilde{\mathbf{x}}(t),\widetilde{\mathbf{u}}_1(t),\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](t)) \right\| \\
&\le C_1\,\|\widehat{\mathbf{x}}(t)-\widetilde{\mathbf{x}}(t)\|
\end{aligned}$$

The Gronwall's inequality (see the appendix in [7]) implies, for every $t\in[\tau',T]$,

$$\|\widehat{\mathbf{x}}(t)-\widetilde{\mathbf{x}}(t)\| \le \left\|\widehat{\mathbf{x}}(\tau')-\widetilde{\mathbf{x}}(\tau')\right\| \exp\left(\int_{\tau'}^t C_1\,\mathrm{d}s\right) \le C\left\|\widehat{\mathbf{x}}(\tau')-\boldsymbol{\xi}'\right\|. \tag{4.29}$$

Now, for every $t\in[\tau',T]$, (4.28) and (4.29) give us

$$\|\widetilde{\mathbf{x}}(t)-\widehat{\mathbf{x}}(t)\| \le C\left(\left\|\widehat{\mathbf{x}}(\tau')-\boldsymbol{\xi}\right\| + \left|\boldsymbol{\xi}-\boldsymbol{\xi}'\right|\right) \le \widetilde{C}\left(\|\boldsymbol{\xi}'-\boldsymbol{\xi}\| + (\tau'-\tau)\right) \tag{4.30}$$

By (4.26) and (4.27), assumption *2.* and *3.* we obtain

$$\begin{aligned}
V^-(\tau',\boldsymbol{\xi}') - V^-(\tau,\boldsymbol{\xi}) &\le \int_{\tau'}^T \left(f(t,\widetilde{\mathbf{x}},\widetilde{\mathbf{u}}_1,\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1]) - f(t,\widehat{\mathbf{x}},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1]) \right)\,\mathrm{d}t + \\
&\qquad - \int_{\tau}^{\tau'} f(t,\widehat{\mathbf{x}},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\widetilde{\mathbf{x}}(T)) - \psi(\widehat{\mathbf{x}}(T)) + 2\epsilon \\
&\le C_1 \int_{\tau'}^T \|\widetilde{\mathbf{x}}(t)-\widehat{\mathbf{x}}(t)\|\,\mathrm{d}t + C_1(\tau'-\tau) + C_2\|\widetilde{\mathbf{x}}(T)-\widehat{\mathbf{x}}(T)\| + 2\epsilon \\
&\le \widehat{C}\left(\|\boldsymbol{\xi}'-\boldsymbol{\xi}\| + (\tau'-\tau)\right) + 2\epsilon.
\end{aligned} \tag{4.31}$$

This concludes the first part of the proof.

Let ϵ again be fixed. Then there exists $\widehat{\Phi}_2 \in \mathcal{S}_2(\tau')$ such that

$$V^-(\tau',\boldsymbol{\xi}') \ge \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau')} \left\{ \int_{\tau'}^T f(t,\mathbf{x},\mathbf{u}_1,\widehat{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\} - \epsilon. \tag{4.32}$$

For every $\mathbf{u}_1 \in \mathcal{U}_1(\tau)$, let us define $\widehat{\mathbf{u}} \in \mathcal{U}_1(\tau')$ by

$$\widehat{\mathbf{u}}_1(t) = \mathbf{u}_1(t), \qquad \forall t\in[\tau',T] \tag{4.33}$$

Fix $\mathbf{u}_{2fix} \in U_2$. Let us define $\widetilde{\Phi}_2 \in \mathcal{S}_2(\tau)$ such that, for every $\mathbf{u}_1 \in \mathcal{U}_1(\tau)$

$$\widetilde{\Phi}_2[\mathbf{u}_1](t) = \begin{cases} \mathbf{u}_{2fix}, & \text{for } t\in[\tau,\tau') \\ \widehat{\Phi}_2[\widehat{\mathbf{u}}_1](t), & \text{for } t\in[\tau',T] \end{cases}$$

Clearly

$$V^-(\tau,\boldsymbol{\xi}) \le \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^T f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\}.$$

Now there exists $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(\tau)$ such that

$$V^-(\tau,\boldsymbol{\xi}) \le \int_\tau^T f(t,\mathbf{x},\widetilde{\mathbf{u}}_1,\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) + \epsilon. \tag{4.34}$$

Now (4.32) gives

$$V^-(\tau',\boldsymbol{\xi}') \ge \int_{\tau'}^T f(t,\mathbf{x},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) - \epsilon. \tag{4.35}$$

where $\widehat{\mathbf{u}}_1$ is defined by $\widetilde{\mathbf{u}}_1$ via relation (4.33). Denoting by $\widetilde{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ the trajectories in (4.34) and in (4.35) respectively, they solve

$$\begin{cases} \dot{\widetilde{\mathbf{x}}}(t) = g(t,\widetilde{\mathbf{x}}(t),\widetilde{\mathbf{u}}_1(t),\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](t)) & \text{in } [\tau,T] \\ \widetilde{\mathbf{x}}(\tau) = \boldsymbol{\xi} \end{cases}$$

and

$$\begin{cases} \dot{\widehat{\mathbf{x}}}(t) = g(t,\widehat{\mathbf{x}}(t),\widehat{\mathbf{u}}_1(t),\widehat{\Phi}_2[\widehat{\mathbf{u}}_1](t)) & \text{in } [\tau',T] \\ \widehat{\mathbf{x}}(\tau') = \boldsymbol{\xi}' \end{cases}$$

By assumption *2.* and using the same arguments of before we obtain inequality (4.30). By (4.34) and (4.35), assumption *2.* and *3.* we obtain

$$\begin{aligned} V^-(\tau',\boldsymbol{\xi}') - V^-(\tau,\boldsymbol{\xi}) &\le \int_{\tau'}^T \Big(f(t,\widetilde{\mathbf{x}},\widetilde{\mathbf{u}}_1,\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1]) - f(t,\widehat{\mathbf{x}},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1]) \Big)\,\mathrm{d}t + \\ &\quad - \int_\tau^{\tau'} f(t,\widetilde{\mathbf{x}},\widetilde{\mathbf{u}}_1,\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1])\,\mathrm{d}t + \psi(\widetilde{\mathbf{x}}(T)) - \psi(\widehat{\mathbf{x}}(T)) + 2\epsilon \\ &\le C_1 \int_{\tau'}^T \|\widetilde{\mathbf{x}}(t) - \widehat{\mathbf{x}}(t)\|\,\mathrm{d}t + C_1(\tau'-\tau) + C_2\|\widetilde{\mathbf{x}}(T) - \widehat{\mathbf{x}}(T)\| + 2\epsilon \\ &\le \widehat{C}\left(\|\boldsymbol{\xi}' - \boldsymbol{\xi}\| + (\tau'-\tau)\right) + 2\epsilon. \end{aligned}$$

This inequality and (4.31) conclude the proof. □

The previous result implies that the lower value function admits the gradient $\nabla V^-(t,\mathbf{x})$ for almost everywhere $(t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n$. Moreover, it gives the possibility to V^- to be a viscosity solution, as we will see in definition 4.7.

Definition 4.7. *Let $H : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $V : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function, with $V(T,\mathbf{x}) = \psi(\mathbf{x})$ in $\mathbb{R}^n$.*
We say that V is a ***viscosity subsolution*** *of the Hamilton–Jacobi equation*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,\mathbf{x}) + H(t,\mathbf{x},\nabla V(t,\mathbf{x})) = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ V(T,\mathbf{x}) = \psi(\mathbf{x}) & \text{in } \mathbb{R}^n \end{cases} \tag{4.36}$$

if whenever v is a test function in $C^\infty((0,T)\times\mathbb{R}^n)$ such that $V - v$ has a local minimum in the point $(t_0,\mathbf{x}_0) \in (0,T)\times\mathbb{R}^n$ we have

$$\frac{\partial v}{\partial t}(t_0,\mathbf{x}_0) + H(t_0,\mathbf{x}_0,\nabla v(t_0,\mathbf{x}_0)) \ge 0. \tag{4.37}$$

We say that V is a ***viscosity supersolution*** *of the equation (4.36) if v is a test function in $C^\infty((0,T)\times\mathbb{R}^n)$ such that $V - v$ has a local maximum in the point $(t_0,\mathbf{x}_0) \in (0,T)\times\mathbb{R}^n$ we have*

$$\frac{\partial v}{\partial t}(t_0,\mathbf{x}_0) + H(t_0,\mathbf{x}_0,\nabla v(t_0,\mathbf{x}_0)) \le 0. \tag{4.38}$$

A function that is both a viscosity subsolution and a viscosity supersolution is called ***viscosity solution****.*

The following Dynamic Programming optimality condition holds (see Theorem 3.1 in [8])

Theorem 4.4. *Let us consider the problem (4.1) with assumptions assumption* 1. *and* 2.. *Then*

$$V^-(\tau,\boldsymbol{\xi}) = \inf_{\Phi_2\in\mathcal{S}_2(\tau)} \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\Phi_2[\mathbf{u}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \right\}$$

for every τ, $\tau+\sigma \in [0,T]$ and $\boldsymbol{\xi} \in \mathbb{R}^n$.

Proof. Let us define the function W by

$$W(\tau,\boldsymbol{\xi}) = \inf_{\Phi_2\in\mathcal{S}_2(\tau)} \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\Phi_2[\mathbf{u}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \right\}$$

for every τ, $\boldsymbol{\xi}$. Let us fix $\epsilon > 0$.

Then there exists a $\widetilde{\Phi}_2 \in \mathcal{S}_2(\tau)$ such that

$$W(\tau,\boldsymbol{\xi}) \geq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \right\} - \epsilon. \tag{4.39}$$

Also, for every $\boldsymbol{\eta} \in \mathbb{R}^n$, by definition of V^-

$$V^-(\tau+\sigma,\boldsymbol{\eta}) = \inf_{\Phi_2\in\mathcal{S}_2(\tau+\sigma)} \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau+\sigma)} \left\{ \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1,\Phi_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\}$$

where $\mathbf{x}$ is the trajectory with initial data $\mathbf{x}(\tau+\sigma) = \boldsymbol{\eta}$. Thus exists a $\widetilde{\Phi}_2^{\boldsymbol{\eta}} \in \mathcal{S}_2(\tau+\sigma)$ such that

$$V^-(\tau+\sigma,\boldsymbol{\eta}) \geq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau+\sigma)} \left\{ \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2^{\boldsymbol{\eta}}[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\} - \epsilon. \tag{4.40}$$

Now define $\overline{\Phi}_2 \in \mathcal{S}_2(\tau)$ in this way: for each $\mathbf{u}_1 \in \mathcal{U}_1(\tau)$ set

$$\overline{\Phi}_2[\mathbf{u}_1](t) = \begin{cases} \widetilde{\Phi}_2[\mathbf{u}_1](t), & \text{for } t\in[\tau,\tau+\sigma] \\ \widetilde{\Phi}_2^{\mathbf{x}(\tau+\sigma)}[\mathbf{u}_1](t), & \text{for } t\in(\tau+\sigma,T] \end{cases}$$

Consequently for any $\mathbf{u}_1 \in \mathcal{U}_1(\tau)$, by (4.39) and (4.40) we have

$$\begin{aligned} W(\tau,\boldsymbol{\xi}) &\geq \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) - \epsilon \\ &\geq \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1,\widetilde{\Phi}_2^{\mathbf{x}(\tau+\sigma)}[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) - 2\epsilon \\ &= \int_\tau^{T} f(t,\mathbf{x},\mathbf{u}_1,\overline{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) - 2\epsilon \end{aligned}$$

So that

$$V^-(\tau,\boldsymbol{\xi}) \leq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{T} f(t,\mathbf{x},\mathbf{u}_1,\overline{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\} \leq W(\tau,\boldsymbol{\xi}) + 2\epsilon. \tag{4.41}$$

Let us pass to the second part of the proof. Now, there exists $\widehat{\Phi}_2 \in \mathcal{S}_2(\tau)$ such that

$$V^-(\tau,\boldsymbol{\xi}) \geq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{T} f(t,\mathbf{x},\mathbf{u}_1,\widehat{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\} - \epsilon. \tag{4.42}$$

Then

$$W(\tau,\boldsymbol{\xi}) \leq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\mathbf{u}_1,\widehat{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \right\}$$

and there exists $\widehat{\mathbf{u}}_1 \in \mathcal{U}_1(\tau)$ such that

$$W(\tau,\boldsymbol{\xi}) \leq \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) + \epsilon. \tag{4.43}$$

For every $\mathbf{u}_1 \in \mathcal{U}_1(\tau+\sigma)$ we define $\mathbf{u}_1^\sharp \in \mathcal{U}_1(\tau)$ by

$$\mathbf{u}_1^\sharp(t) = \begin{cases} \widehat{\mathbf{u}}_1(t), & \text{for } t\in[\tau,\tau+\sigma] \\ \mathbf{u}_1(t), & \text{for } t\in(\tau+\sigma,T] \end{cases}$$

and we define $\Phi_2^\dagger \in \mathcal{S}_2(\tau+\sigma)$ by

$$\Phi_2^\dagger[\mathbf{u}_1](t) = \widehat{\Phi}_2[\mathbf{u}_1^\sharp](t), \qquad \forall \mathbf{u}_1\in\mathcal{U}_1(\tau+\sigma),\ t\in[\tau+\sigma,T]$$

Hence

$$V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \leq \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau+\sigma)} \left\{ \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1,\Phi_2^\dagger[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\}.$$

Clearly there exists $\mathbf{u}_1^\dagger \in \mathcal{U}_1(\tau+\sigma)$ such that

$$V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) \leq \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1^\dagger,\Phi_2^\dagger[\mathbf{u}_1^\dagger])\,\mathrm{d}t + \psi(\mathbf{x}(T)) + \epsilon. \tag{4.44}$$

Now we define $\mathbf{u}_1' \in \mathcal{U}_1(\tau)$ by

$$\mathbf{u}_1'(t) = \begin{cases} \widehat{\mathbf{u}}_1(t), & \text{for } t \in [\tau, \tau+\sigma] \\ \mathbf{u}_1^\dagger(t), & \text{for } t \in (\tau+\sigma, T] \end{cases}$$

Now by (4.43) and (4.44) we have

$$\begin{aligned} W(\tau,\boldsymbol{\xi}) &\le \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + V^-(\tau+\sigma,\mathbf{x}(\tau+\sigma)) + \epsilon \\ &\le \int_\tau^{\tau+\sigma} f(t,\mathbf{x},\widehat{\mathbf{u}}_1,\widehat{\Phi}_2[\widehat{\mathbf{u}}_1])\,\mathrm{d}t + \int_{\tau+\sigma}^{T} f(t,\mathbf{x},\mathbf{u}_1^\dagger,\Phi_2^\dagger[\mathbf{u}_1^\dagger])\,\mathrm{d}t + \psi(\mathbf{x}(T)) + 2\epsilon \\ &= \int_\tau^{T} f(t,\mathbf{x},\mathbf{u}_1',\widehat{\Phi}_2[\mathbf{u}_1'])\,\mathrm{d}t + \psi(\mathbf{x}(T)) + 2\epsilon \\ &\le \sup_{\mathbf{u}_1\in\mathcal{U}_1(\tau)} \left\{ \int_\tau^{T} f(t,\mathbf{x},\mathbf{u}_1,\widehat{\Phi}_2[\mathbf{u}_1])\,\mathrm{d}t + \psi(\mathbf{x}(T)) \right\} + 2\epsilon \\ &\le V^-(\tau,\boldsymbol{\xi}) + 3\epsilon. \end{aligned}$$

This last inequality and (4.41) conclude the proof. □

And now we are in the position to give the main property for the two values functions V^- and V^+:

Theorem 4.5. *Let us consider the problem (4.1) with the assumptions* 1.– 2.. *Then*

A. V^- is a viscosity solution for

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,\mathbf{x}) + H^-_{DP}(t,\mathbf{x},\nabla_{\mathbf{x}} V(t,\mathbf{x})) = 0 & \textit{for } (t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n \\ V(T,\mathbf{x}) = \psi(\mathbf{x}) & \textit{for } \mathbf{x}\in\mathbb{R}^n \end{cases} \tag{4.45}$$

B. V^- is the unique viscosity solution for (4.45);

A'. V^+ is a viscosity solution for

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,\mathbf{x}) + H^+_{DP}(t,\mathbf{x},\nabla_{\mathbf{x}} V(t,\mathbf{x})) = 0 & \textit{for } (t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n \\ V(T,\mathbf{x}) = \psi(\mathbf{x}) & \textit{for } \mathbf{x}\in\mathbb{R}^n \end{cases} \tag{4.46}$$

B'. V^+ is the unique viscosity solution for (4.46).

The system in (4.46) is called **upper Isaacs' equation**, while (4.45) is called **lower Isaacs' equation**. The proof of A'. is very similar to the proof of A. The proofs of B. and B'. are similar but they require a comparison principle for BHJ equation: such very interesting argument is very difficult and long, and it requires another course.

Proof of A. It is obvious, by definition, that $V^-(T,\mathbf{x}) = \psi(\mathbf{x})$, for every $\mathbf{x}\in\mathbb{R}^n$. So, let us fix $(t_0,\mathbf{x}_0)\in(0,T)\times\mathbb{R}^n$.
First part of the proof: V^- is a supersolution. Let $v\in C^1((0,T)\times\mathbb{R}^n)$ be a test function touching V^- from below at $(t_0,\mathbf{x}_0)$, i.e.

$$V^-(t_0,\mathbf{x}_0) = v(t_0,\mathbf{x}_0) \quad \text{and } V^-(t,\mathbf{x}) \ge v(t,\mathbf{x}) \quad \text{in a neighborhood of } (t_0,\mathbf{x}_0). \tag{4.47}$$

We have to prove that

$$\frac{\partial v}{\partial t}(t_0,\mathbf{x}_0) + H^-_{DP}(t_0,\mathbf{x}_0,\nabla v(t_0,\mathbf{x}_0)) \le 0.$$

Let us assume that this is not true and that there exists $\theta > 0$ such that

$$\frac{\partial v}{\partial t}(t_0,\mathbf{x}_0) + H^-_{DP}(t_0,\mathbf{x}_0,\nabla v(t_0,\mathbf{x}_0)) \ge \theta. \tag{4.48}$$

Defining the function Γ in a compact neighborhood of $(t_0,\mathbf{x}_0)$ by

$$\Gamma(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) = \frac{\partial v}{\partial t}(t,\mathbf{x}) + f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) + \nabla v(t,\mathbf{x})\cdot g(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)$$

(4.48) is equivalent to

$$\max_{\mathbf{u}_1\in U_1}\ \min_{\mathbf{u}_2\in U_2} \Gamma(t_0,\mathbf{x}_0,\mathbf{u}_1,\mathbf{u}_2) \ge \theta$$

Hence there exists $\mathbf{u}_1^* \in U_1$ such that

$$\min_{\mathbf{u}_2 \in U_2} \Gamma(t_0, \mathbf{x}_0, \mathbf{u}_1^*, \mathbf{u}_2) \geq \theta$$

Since Γ is uniformly continuous in its domain and by assumption *2.* on g, there exists $\tau > 0$ such that

$$\min_{\mathbf{u}_2 \in U_2} \Gamma(s, \widetilde{\mathbf{x}}(s), \mathbf{u}_1^*, \mathbf{u}_2) \geq \frac{\theta}{2}, \qquad \forall s \in [t_0, t_0 + \tau] \tag{4.49}$$

where $\widetilde{\mathbf{u}}_1$ and $\widetilde{\mathbf{u}}_2$ are generic controls in $\mathcal{U}_1(t_0)$ and in $\mathcal{U}_2(t_0)$ respectively, and $\widetilde{\mathbf{x}}$ solves

$$\begin{cases} \dot{\widetilde{\mathbf{x}}} = g(t, \widetilde{\mathbf{x}}, \widetilde{\mathbf{u}}_1, \widetilde{\mathbf{u}}_2) & \text{in } [t_0, T] \\ \widetilde{\mathbf{x}}(t_0) = \mathbf{x}_0 \end{cases} \tag{4.50}$$

Hence, choosing $\widetilde{\mathbf{u}}_1(\cdot) = \mathbf{u}_1^*$ and for any $\Phi_2 \in \mathcal{S}_2(t_0)$ we have that inequality (4.49) implies

$$\Gamma(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}_1(s), \Phi_2[\widetilde{\mathbf{u}}_1](s)) \geq \frac{\theta}{2}, \qquad \forall s \in [t_0, t_0 + \tau]$$

where now $\widetilde{\mathbf{x}}$ is the trajectory in (4.50) associated to the controls $\widetilde{\mathbf{u}}_1 = \mathbf{u}_1^*$ and $\Phi_2[\widetilde{\mathbf{u}}_1]$. If we integrate the last inequality, we obtain that there exists $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(t_0)$ such that for every $\Phi_2 \in \mathcal{S}_2(t_0)$ we have

$$\int_{t_0}^{t_0+\tau} \Gamma(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}_1(s), \Phi_2[\widetilde{\mathbf{u}}_1](s)) ds \geq \frac{\tau\theta}{2}$$

and hence

$$\inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u} \in \mathcal{U}_1(t_0)} \int_{t_0}^{t_0+\tau} \left(\frac{\partial v}{\partial t}(s, \mathbf{x}) + f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) + \nabla v(s, \mathbf{x}) \cdot g(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \right) ds \geq \frac{\tau\theta}{2} \tag{4.51}$$

where $\mathbf{x}$ solves

$$\begin{cases} \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) & \text{in } [t_0, T] \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{4.52}$$

Now by Theorem 4.4 we know that

$$V^-(t_0, \mathbf{x}_0) = \inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \left\{ \int_{t_0}^{t_0+\tau} f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \, \mathrm{d}s + V^-(t_0 + \tau, \mathbf{x}(t_0 + \tau)) \right\} \tag{4.53}$$

with $\mathbf{x}$ as before. For every such $\mathbf{x}$, requirement (4.47) and the lipschitz assumption *2.* on g imply that, for τ small enough

$$0 = V^-(t_0, \mathbf{x}_0) - v(t_0, \mathbf{x}_0) \leq V^-(t_0 + \tau, \mathbf{x}(t_0 + \tau)) - v(t_0 + \tau, \mathbf{x}(t_0 + \tau)) \tag{4.54}$$

Since $\mathbf{x}$ is continuous and v is in C^1, (4.52) implies

$$\begin{aligned} v(t_0 + \tau, \mathbf{x}(t_0 + \tau)) - v(t_0, \mathbf{x}_0) &= \int_{t_0}^{t_0+\tau} \frac{dv(s, \mathbf{x}(s))}{ds} ds \\ &\int_{t_0}^{t_0+\tau} \left(\frac{\partial v}{\partial t}(s, \mathbf{x}(s)) + \nabla v(s, \mathbf{x}(s)) \cdot g(s, \mathbf{x}(s), \mathbf{u}_1(s), \Phi_2[\mathbf{u}_1](s)) \right) ds \end{aligned} \tag{4.55}$$

Relations (4.53)–(4.55) give

$$\begin{aligned} 0 \quad &\geq \quad \inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \left\{ \int_{t_0}^{t_0+\tau} f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}]) \, \mathrm{d}s + v(t_0 + \tau, \mathbf{x}(t_0 + \tau)) - v(t_0, \mathbf{x}_0) \right\} \\ &= \quad \inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \int_{t_0}^{t_0+\tau} \left(f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) + \frac{\partial \psi}{\partial t}(s, \mathbf{x}) + \nabla v(s, \mathbf{x}) \cdot g(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \right) \mathrm{d}s \end{aligned}$$

This inequality contradicts (4.51): hence (4.48) is false and this concludes the first part of the proof.

Second part of the proof: V^- is a subsolution. Now, let $v \in C^1((0,T) \times \mathbb{R}^n)$ be a test function touching V^- from above at $(t_0, \mathbf{x}_0)$, i.e.

$$V^-(t_0, \mathbf{x}_0) = v(t_0, \mathbf{x}_0) \quad \text{and } V^-(t, \mathbf{x}) \leq v(t, \mathbf{x}) \quad \text{in a neighborhood of } (t_0, \mathbf{x}_0). \tag{4.56}$$

We have to prove that

$$\frac{\partial v}{\partial t}(t_0, \mathbf{x}_0) + H_{DP}^-(t_0, \mathbf{x}_0, \nabla v(t_0, \mathbf{x}_0)) \geq 0.$$

Let us assume that this is not true and that there exists $\theta > 0$ such that

$$\frac{\partial v}{\partial t}(t_0, \mathbf{x}_0) + H_{DP}^-(t_0, \mathbf{x}_0, \nabla v(t_0, \mathbf{x}_0)) \geq -\theta. \tag{4.57}$$

This is equivalent to

$$\max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \Gamma(t_0, \mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) \leq -\theta$$

Hence, for every $\mathbf{u}_1 \in U_1$ there exists $\mathbf{u}_2^{\mathbf{u}_1} \in U_2$ such that

$$\Gamma(t_0, \mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2^{\mathbf{u}_1}) \leq -\theta$$

Since Γ is uniformly continuous we have

$$\Gamma(t_0, \mathbf{x}_0, \widetilde{\mathbf{u}}_1, \mathbf{u}_2^{\mathbf{u}_1}) \leq -\frac{3\theta}{4} \tag{4.58}$$

for every $\mathbf{u}_1 \in U_1$, $\widetilde{\mathbf{u}}_1 \in B_{\mathbb{R}^{k_1}}(\mathbf{u}_1, r(\mathbf{u}_1)) \cap U_1$ and for some $r(\mathbf{u}_1) > 0$. Since U_1 is compact (see assumption *1.*) there exist finitely many distinct points $\{\mathbf{u}_1^i\}_{i=1}^N \subset U_1$, $\{\mathbf{u}_2^{\mathbf{u}_1^i}\}_{i=1}^N \subset U_2$ and rays $\{r(\mathbf{u}_1^i)\}_{i=1}^N$ such that

$$U_1 \subset \bigcup_{i=1}^N B_{\mathbb{R}^{k_1}}(\mathbf{u}_1^i, r(\mathbf{u}_1^i))$$

and

$$\Gamma(t_0, \mathbf{x}_0, \widetilde{\mathbf{u}}_1, \mathbf{u}_2^{\mathbf{u}_1^i}) \leq -\frac{3\theta}{4}, \qquad \forall \widetilde{\mathbf{u}}_1 \in B_{\mathbb{R}^2}(\mathbf{u}_1^i, r(\mathbf{u}_1^i)) \cap U_1$$

Let us define $\phi : U_1 \to U_2$ by $\phi(\mathbf{u}_1) = \mathbf{u}_2^{\mathbf{u}_1^j}$ with $j = j(\mathbf{u}_1)$ such that

$$\mathbf{u}_1 \in B_{\mathbb{R}^{k_1}}(\mathbf{u}_1^j, r(\mathbf{u}_1^j)) \setminus \bigcup_{i=1}^{j-1} B_{\mathbb{R}^{k_1}}(\mathbf{u}_1^i, r(\mathbf{u}_1^i)).$$

Hence (4.58) implies

$$\Gamma(t_0, \mathbf{x}_0, \mathbf{u}_1, \phi(\mathbf{u}_1)) \leq -\frac{3\theta}{4}$$

for every $\mathbf{u}_1 \in U_1$. Since Γ is uniformly continuous there exists $\tau > 0$ such that we have

$$\Gamma(s, \widetilde{\mathbf{x}}(s), \mathbf{u}_1, \phi(\mathbf{u}_1)) \leq -\frac{\theta}{2}, \qquad \forall s \in [t_0, t_0 + \tau] \tag{4.59}$$

for every $\mathbf{u}_1 \in U_1$ and for every $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(t_0)$, $\widetilde{\mathbf{u}}_2 \in \mathcal{U}_2(t_0)$ where $\widetilde{\mathbf{x}}$ is the associated trajectory as in (4.50). Now let us define $\widetilde{\Phi}_2 \in \mathcal{S}_2(t_0)$ such that

$$\widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](s) = \phi(\widetilde{\mathbf{u}}_1(s)), \qquad \forall \widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(t_0), \ s \in [t_0, T]$$

Using (4.59)

$$\Gamma(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}_1(s), \widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](s)) \leq -\frac{\theta}{2}, \qquad \forall s \in [t_0, t_0 + \tau]$$

for every $\widetilde{\mathbf{u}}_1 \in \mathcal{U}_1(t_0)$ and where $\widetilde{\mathbf{x}}$ is as in (4.50) with $\mathbf{u}_2 = \Phi_2[\mathbf{u}_1]$. If we integrate the last inequality, we obtain that there exists $\widetilde{\Phi}_2 \in \mathcal{S}_2(t_0)$ such that for every $\widetilde{\mathbf{u}}_1 \in \mathcal{U}(t_0)$ we have

$$\int_{t_0}^{t_0+\tau} \Gamma(s, \widetilde{\mathbf{x}}(s), \widetilde{\mathbf{u}}_1(s), \widetilde{\Phi}_2[\widetilde{\mathbf{u}}_1](s)) ds \leq -\frac{\tau\theta}{2}$$

and hence

$$\inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \int_{t_0}^{t_0+\tau} \left(\frac{\partial v}{\partial t}(s, \mathbf{x}) + f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) + \nabla v(s, \mathbf{x}) \cdot g(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \right) ds \leq -\frac{\tau\theta}{2} \tag{4.60}$$

where $\mathbf{x}$ is as in (4.52). For every such $\mathbf{x}$, requirement (4.56) and the lipschitz assumption *2.* on g imply that, for τ small enough

$$0 = V^-(t_0, \mathbf{x}_0) - v(t_0, \mathbf{x}_0) \geq V^-(t_0 + \tau, \mathbf{x}(t_0 + \tau)) - v(t_0 + \tau, \mathbf{x}(t_0 + \tau)) \tag{4.61}$$

Relations (4.53), (4.55) and (4.61) give

$$\begin{aligned} 0 \quad &\leq \quad \inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \left\{ \int_{t_0}^{t_0+\tau} f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \, \mathrm{d}s + v(t_0 + \tau, \mathbf{x}(t_0 + \tau)) - v(t_0, \mathbf{x}_0) \right\} \\ &= \quad \inf_{\Phi_2 \in \mathcal{S}_2(t_0)} \sup_{\mathbf{u}_1 \in \mathcal{U}_1(t_0)} \int_{t_0}^{t_0+\tau} \left(f(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) + \frac{\partial v}{\partial t}(s, \mathbf{x}) + \nabla v(s, \mathbf{x}) \cdot g(s, \mathbf{x}, \mathbf{u}_1, \Phi_2[\mathbf{u}_1]) \right) \, \mathrm{d}s \end{aligned}$$

This inequality contradicts (4.60): hence (4.57) is false and this concludes the proof.

□

4.2.4 Isaacs' condition, Isaacs' equation and value function

If the Isaacs' condition is satisfied, clearly the systems (4.46) and (4.45) coincide, and we obtain

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,\mathbf{x}) + H_{DP}(t,\mathbf{x},\nabla_{\mathbf{x}} V(t,\mathbf{x})) = 0 & \text{for } (t,\mathbf{x}) \in [0,T]\times\mathbb{R}^n \\ V(T,\mathbf{x}) = \psi(\mathbf{x}) & \text{for } \mathbf{x}\in\mathbb{R}^n \end{cases} \tag{4.62}$$

The previous system (4.62) is called **Isaacs' equation**.
The fundamental theorem on the value function is the following

Theorem 4.6. *Let us consider the problem (4.1) with the assumptions* 1. *and* 2. *Let us suppose that the Isaacs' condition (4.23) holds. Then*

a. the problem (4.1) has value function V, i.e. (4.16) is always an equality;

b. V is the unique viscosity solution of (4.62);

Proof. The proof of *a.* and *b.* of this theorem is an easy consequence of Theorem 4.5 with the Isaacs's condition.
□

As we said at the beginning of this section, we have given the results of this section for open-loop strategies but they are also true for more general problems and feedback strategies. The main result is in the previous theorem: if the Isaacs' condition (4.23) holds, then the problem has value function and satisfies the Isaacs' equation (4.62).

4.3 Regular solutions of Isaacs' equation for general problems

In the previous section we discussed the details of the upper and lower value functions, their regularity and relations with respect to the upper e lower Isaacs' equation and with open-loop strategies. In this section we are interested to study the case where the Isaacs' condition holds, the feedback strategies are considered and the value function exists and it is regular.

Definition 4.8. *Let us consider the problem (4.1). The* ***lower value function*** $V^- : [0,T]\times\mathbb{R}^n \to [-\infty,+\infty]$ *is defined by*

$$V^-(\tau,\boldsymbol{\xi}) = \inf_{\boldsymbol{\nu}_2}\sup_{\boldsymbol{\nu}_1} \int_\tau^T f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,\mathrm{d}t + \psi(\mathbf{x}(T)), \tag{4.63}$$

where $(\mathbf{u}_1,\mathbf{u}_2)\in\mathcal{A}_{FB}$, with $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t,\mathbf{x}(t))$ for $i=1,2$, is admissible with trajectory $\mathbf{x}$ unique solution of

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t,\mathbf{x}(t),\boldsymbol{\nu}_1(t,\mathbf{x}(t)),\boldsymbol{\nu}_2(t,\mathbf{x}(t))) & \textit{for a.e. } t\in[\tau,T] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

Similarly, the ***upper value function*** $V^+ : [0,T]\times\mathbb{R}^n \to [-\infty,+\infty]$ *is defined by*

$$V^+(\tau,\boldsymbol{\xi}) = \sup_{\boldsymbol{\nu}_1}\inf_{\boldsymbol{\nu}_2} \int_\tau^T f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,\mathrm{d}t + \psi(\mathbf{x}(T)). \tag{4.64}$$

Note that V^- and V^+ admits the values $\pm\infty$ since now we have no particular assumptions on our problem (4.1).

As in the open-loop strategies case and for the same reasons (see (4.16)), for every $(\tau,\boldsymbol{\xi})$ we have $V^-(\tau,\boldsymbol{\xi}) \le V^+(\tau,\boldsymbol{\xi})$. We say, as in Definition 4.4, that the problem (4.1) admits value V function if

$$V(\tau,\boldsymbol{\xi}) = V^-(\tau,\boldsymbol{\xi}) = V^+(\tau,\boldsymbol{\xi})$$

.

In the spirit of Theorem A.6, we have the following result (see for example Corollary 6.6 and Theorem 8.1 in [2]):

Theorem 4.7. *Let us consider the problem (4.1) with f, g and ψ continuous. Let us suppose that the Isaacs' condition (4.23) is satisfied. Let $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 solution of the Isaacs' equation (4.62). Let $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$, with the corresponding trajectory $\mathbf{x}^*$ with $\mathbf{x}^*(0) = \boldsymbol{\alpha}$, be such that*

$$\begin{aligned} -\frac{\partial V}{\partial t}(t,\mathbf{x}) &= f(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) + \nabla_{\mathbf{x}} V(t,\mathbf{x}) \cdot g(t,\mathbf{x},\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x})) \\ &= \min_{\mathbf{v}_2 \in U_2} \max_{\mathbf{v}_1 \in U_1} \Big(f(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) + \nabla_{\mathbf{x}} V(t,\mathbf{x}) \cdot g(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) \Big) \qquad (4.65) \\ &= \max_{\mathbf{v}_1 \in U_1} \min_{\mathbf{v}_2 \in U_2} \Big(f(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) + \nabla_{\mathbf{x}} V(t,\mathbf{x}) \cdot g(t,\mathbf{x},\mathbf{v}_1,\mathbf{v}_2) \Big) \end{aligned}$$

Then $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a Nash equilibrium for the game (4.1) in the class of feedback strategy. In particular

$$V^-(0,\boldsymbol{\alpha}) = V^+(0,\boldsymbol{\alpha}) = J(\mathbf{u}_1^*, \mathbf{u}_2^*). \qquad (4.66)$$

Moreover, if for every initial data $(\tau, \boldsymbol{\xi}) \in [0,T] \times \mathbb{R}^n$ there exists the corresponding trajectory $\mathbf{x}$, solution of

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t,\mathbf{x}(t),\boldsymbol{\nu}_1^*(t,\mathbf{x}(t)),\boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) & \text{a.e. in } [\tau, T] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

then V is the value function for (4.1).

We remark that (4.65) implies that $(\boldsymbol{\nu}_1^*(t,\mathbf{x}),\boldsymbol{\nu}_2^*(t,\mathbf{x}))$ realizes the max-min, for every $(t,\mathbf{x}) \in [0,T] \times \mathbb{R}^n$. In particular, along the optimal trajectory of the problem (4.1) we have

$$-\frac{\partial V}{\partial t}(t,\mathbf{x}^*(t)) = f(t,\mathbf{x}^*(t),\mathbf{u}_1^*(t),\mathbf{u}_2^*(t)) + \nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) \cdot g(t,\mathbf{x}^*(t),\mathbf{u}_1^*(t),\mathbf{u}_2^*(t)),$$

for every $t \in [0,T]$.

Proof of Theorem 4.7. Let $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(t, \mathbf{x}^*(t))$, with the corresponding trajectory $\mathbf{x}^*$ with $\mathbf{x}^*(0) = \boldsymbol{\alpha}$. For every fixed t,

$$\begin{aligned} \frac{\partial V}{\partial t}(t,\mathbf{x}^*(t)) &= -\max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \Big(f(t,\mathbf{x}^*(t),\mathbf{u}_1,\mathbf{u}_2) + \nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) \cdot g(t,\mathbf{x}^*(t),\mathbf{u}_1,\mathbf{u}_2) \Big) \\ &= -f(t,\mathbf{x}^*(t),\boldsymbol{\nu}_1^*(t,\mathbf{x}^*(t)),\boldsymbol{\nu}_2^*(t,\mathbf{x}^*(t))) - \nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) \cdot g(t,\mathbf{x}^*(t),\boldsymbol{\nu}_1^*(t,\mathbf{x}^*(t)),\boldsymbol{\nu}_2^*(t,\mathbf{x}^*(t))) \\ &= -f(t,\mathbf{x}^*(t),\mathbf{u}_1^*(t),\mathbf{u}_2^*(t)) - \nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t) \qquad (4.67) \end{aligned}$$

Since V is differentiable, the fundamental theorem of integral calculus implies

$$\begin{aligned} V(T,\mathbf{x}^*(T)) - V(0,\mathbf{x}^*(0)) &= \int_0^T \frac{\mathrm{d}V(t,\mathbf{x}^*(t))}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_0^T \frac{\partial V}{\partial t}(t,\mathbf{x}^*(t)) + \nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t)\,\mathrm{d}t \\ \text{(by (4.67))} \quad &= -\int_0^T f(t,\mathbf{x}^*(t),\mathbf{u}_1^*(t),\mathbf{u}_2^*(t))\,\mathrm{d}t. \qquad (4.68) \end{aligned}$$

Let $(\mathbf{u}_1, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_1(t) = \boldsymbol{\nu}_1(t,\mathbf{x}(t))$ and $\mathbf{u}_2^*(t) = \boldsymbol{\nu}_2^*(t,\mathbf{x}(t))$, with the corresponding trajectory $\mathbf{x}$ with $\mathbf{x}(0) = \boldsymbol{\alpha}$. For every fixed t,

$$\begin{aligned} \frac{\partial V}{\partial t}(t,\mathbf{x}(t)) &= -\max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \Big(f(t,\mathbf{x}(t),\mathbf{u}_1,\mathbf{u}_2) + \nabla_{\mathbf{x}} V(t,\mathbf{x}(t)) \cdot g(t,\mathbf{x}(t),\mathbf{u}_1,\mathbf{u}_2) \Big) \\ &= -\max_{\mathbf{u}_1 \in U_1} \Big(f(t,\mathbf{x}(t),\mathbf{u}_1,\boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) + \nabla_{\mathbf{x}} V(t,\mathbf{x}(t)) \cdot g(t,\mathbf{x}(t),\mathbf{u}_1,\boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) \Big) \\ &\le -f(t,\mathbf{x}(t),\boldsymbol{\nu}_1(t,\mathbf{x}(t)),\boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) - \nabla_{\mathbf{x}} V(t,\mathbf{x}(t)) \cdot g(t,\mathbf{x}(t),\boldsymbol{\nu}_1(t,\mathbf{x}(t)),\boldsymbol{\nu}_2^*(t,\mathbf{x}(t))) \\ &= -f(t,\mathbf{x}(t),\mathbf{u}_1(t),\mathbf{u}_2^*(t)) - \nabla_{\mathbf{x}} V(t,\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \qquad (4.69) \end{aligned}$$

Again we have

$$\begin{aligned} V(T,\mathbf{x}(T)) - V(0,\mathbf{x}(0)) &= \int_0^T \frac{\mathrm{d}V(t,\mathbf{x}(t))}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_0^T \frac{\partial V}{\partial t}(t,\mathbf{x}(t)) + \nabla_{\mathbf{x}} V(t,\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)\,\mathrm{d}t \\ \text{(by (4.69))} \quad &\le -\int_0^T f(t,\mathbf{x}(t),\mathbf{u}_1(t),\mathbf{u}_2^*(t))\,\mathrm{d}t. \end{aligned} \tag{4.70}$$

We remark that $\mathbf{x}^*(0) = \mathbf{x}(0) = \boldsymbol{\alpha}$; if we subtract the two expressions in (4.68) and in (4.70), then we obtain

$$V(T,\mathbf{x}^*(T)) - V(T,\mathbf{x}(T)) \ge -\int_0^T f(t,\mathbf{x}^*,\mathbf{u}_1^*,\mathbf{u}_2^*)\,\mathrm{d}t + \int_0^T f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2^*)\,\mathrm{d}t.$$

Using the final condition in the Isaacs' equation (4.62), the previous inequality becomes

$$J(\mathbf{u}_1^*,\mathbf{u}_2^*) = \int_0^T f(t,\mathbf{x}^*,\mathbf{u}_1^*,\mathbf{u}_2^*)\,\mathrm{d}t + \psi(\mathbf{x}^*(T)) \ge \int_0^T f(t,\mathbf{x},\mathbf{u}_1,\mathbf{u}_2^*)\,\mathrm{d}t + \psi(\mathbf{x}(T)) = J(\mathbf{u}_1,\mathbf{u}_2^*),$$

for every $(\mathbf{u}_1,\mathbf{u}_2^*) \in \mathcal{A}_{FB}$. A similar argument proves that

$$J(\mathbf{u}_1^*,\mathbf{u}_2^*) \le J(\mathbf{u}_1^*,\mathbf{u}_2),$$

for every $(\mathbf{u}_1^*,\mathbf{u}_2) \in \mathcal{A}_{FB}$. Hence we have that $(\mathbf{u}_1^*,\mathbf{u}_2^*)$ is a Nash equilibrium in the family of feedback strategies.

Now, it is obvious that,

$$\sup_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2) \ge J(\mathbf{u}_1^*,\mathbf{u}_2),$$

for every $\mathbf{u}_2$; hence

$$\inf_{\mathbf{u}_2}\sup_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2) \ge \inf_{\mathbf{u}_2} J(\mathbf{u}_1^*,\mathbf{u}_2).$$

Now, the previous inequality, the definition of lower value function V^- in (4.63) and the fact that $(\mathbf{u}_1^*,\mathbf{u}_2^*)$ is a Nash equilibrium imply

$$V^-(0,\boldsymbol{\alpha}) = \inf_{\mathbf{u}_2}\sup_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2) \ge \inf_{\mathbf{u}_2} J(\mathbf{u}_1^*,\mathbf{u}_2) = J(\mathbf{u}_1^*,\mathbf{u}_2^*).$$

A similar argument gives $V^+(0,\boldsymbol{\alpha}) \le J(\mathbf{u}_1^*,\mathbf{u}_2^*)$. Relation (4.16) gives (4.66).

Finally, if we replace the initial data $\mathbf{x}(0) = \boldsymbol{\alpha}$ in the game with the new initial data $\mathbf{x}(\tau) = \boldsymbol{\xi}$, then the same proof gives that the game has value function V. □

In the assumption of Theorem 4.7 we have that

$$V(0,\boldsymbol{\alpha}) = J(\mathbf{u}_1^*,\mathbf{u}_2^*) \tag{4.71}$$

Lewin devotes the section 3.2 of his book [12] to the definitions of the optimal strategies and the value function of a differential game using, as a matter of fact, equality (4.71) to define the value function.

Clearly, in the assumptions of Theorem 4.1 and Theorem 4.7, we know that

$$\nabla_{\mathbf{x}} V(t,\mathbf{x}^*(t)) = \boldsymbol{\lambda}^*(t), \tag{4.72}$$

for every $t \in [0,T]$.

A geometric proof of Isaacs' equation as necessary condition

We are interested to give a very different proof of the Isaacs' equation (4.62), based on geometric ideas. In particular, let us consider a different version of the problem (4.1) with $f = 0$ and T free: more precisely let us consider

$$\begin{cases} \text{Player I: } \max_{\mathbf{u}_1} J(\mathbf{u}_1, \mathbf{u}_2), \qquad \text{Player II: } \min_{\mathbf{u}_2} J(\mathbf{u}_1, \mathbf{u}_2) \\ J(\mathbf{u}_1, \mathbf{u}_2) = \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{x}(0) = \boldsymbol{\alpha} \\ (T, \mathbf{x}(T)) \in \partial\mathcal{T} \end{cases} \tag{4.73}$$

where $\mathcal{T} \subset [0, \infty) \times \mathbb{R}^n$ is a closed target set and $\mathcal{G}$ is the game set, i.e. if $(\tau, \boldsymbol{\xi}) \in \mathcal{G}$ then there exists a trajectory $\mathbf{x}$ which transfers the initial point $(\tau, \boldsymbol{\xi})$ in a point $(T, \mathbf{x}(T))$ with $\mathbf{x}(T) \in \partial\mathcal{T}$ (let us recall that $\mathcal{T} \subset \mathcal{G}$). As usual we define the exit time $T_{\mathbf{x}}$ for the trajectory $\mathbf{x}$ by (since $\mathcal{T}$ is closed)

$$T_{\mathbf{x}} = \inf\{t \geq 0 : (t, \mathbf{x}(t)) \in \mathcal{T}\} = \inf\{t \geq 0 : (t, \mathbf{x}(t)) \in \partial\mathcal{T}\}. \tag{4.74}$$

Theorem 4.8. *Let us consider the problem (4.73) with g and ψ continuous, with a closed target set $\mathcal{T}$ and game set $\mathcal{G}$. Let $(0, \boldsymbol{\alpha}) \in \mathcal{G} \setminus \mathcal{T}$. Let us suppose that*

i. the Isaacs' condition holds; in this case we have

$$H_{DP}(t, \mathbf{x}, \boldsymbol{\lambda}) = \max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = \min_{\mathbf{u}_2 \in U_2} \max_{\mathbf{u}_1 \in U_1} \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}_1, \mathbf{u}_2); \tag{4.75}$$

ii. the problem has value function V, with $V \in C^1(\mathcal{G} \setminus \mathcal{T})$ and $\nabla V(t, \mathbf{x}) = (\frac{\partial V}{\partial t}(t, \mathbf{x}), \nabla_{\mathbf{x}} V(t, \mathbf{x})) \neq 0$ for all $(t, \mathbf{x}) \in \mathcal{G} \setminus \mathcal{T}$.

Let us consider a Nash equilibrium $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ for the problem (4.73) and its optimal trajectory $\mathbf{x}^*$ with exit time T^*. Then V satisfies the Isaacs's equation (4.62) along the optimal path; more precisely,

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + H_{DP}(t, \mathbf{x}^*(t), \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t))) = 0 \quad \text{for } t \in [0, T^*) \\ V(T^*, \mathbf{x}(T^*)) = \psi(T^*, \mathbf{x}(T^*)) \end{cases} \tag{4.76}$$

Proof. Let us consider a Nash equilibrium $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ and its optimal trajectory $\mathbf{x}^*$ with exit time T^* for the problem (4.73): clearly we have

$$\mathbf{x}^*(0) = \boldsymbol{\alpha}, \qquad (T^*, \mathbf{x}^*(T^*)) \in \partial\mathcal{T}.$$

The final condition in (4.76) is obvious. Moreover, for every $\tau \in [0, T^*]$ fixed, if we consider the new problem (4.73) with the new initial data

$$\mathbf{x}(\tau) = \mathbf{x}^*(\tau),$$

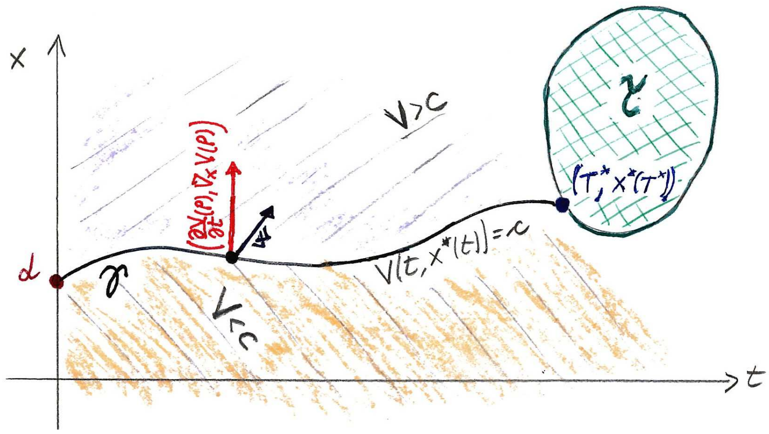
the new optimal trajectory coincides, by the Bellman's principle, with $\mathbf{x}^*$ (the idea of the proof coincides with the classical situation of an optimal control problem). Hence, for every $\tau \in [0, T^*]$,

$$V(\tau, \mathbf{x}^*(\tau)) = \psi(T^*, \mathbf{x}^*(T^*)) = c$$

where c is a constant. Clearly we obtain

$$\frac{\mathrm{d}V(t, \mathbf{x}^*(t))}{\mathrm{d}t} = \frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t) = 0, \tag{4.77}$$

for every $t \in [0, T^*)$.
Let us consider the curve $\gamma : [0, T^*] \to \mathcal{G}$, defined by $\gamma(t) = (t, \mathbf{x}^*(t))$: it is such that $V(\gamma(t)) = c$ for every $t \in [0, T^*]$. Since $\nabla V(t, \mathbf{x}) \neq 0$, the Dini's theorem guarantees that locally the curve γ divides the set $\mathcal{G} \setminus \mathcal{T}$ in two different regions where $V(t, \mathbf{x}) > c$ and $V(t, \mathbf{x}) < c$.



Now fix $t \in [0, T^*)$ and consider the point $P = (t, \mathbf{x}^*(t))$. In such point P, the function V has the maximum growth in the direction of the vector

$$\nabla V(\gamma(t)) = \left(\frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)),\ \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \right).$$

Since the Player I wants to maximizes, he wishes to move P in such direction; but he has some "constraints" for the movements of the point P, i.e. on the trajectory, given by the dynamics and the choice of the Player II. Hence, if the Player II choices $\mathbf{u}_2^*(t)$, then Player I considers its strategy such that

$$\mathbf{u}_1^*(t) \in \arg\max_{\mathbf{u}_1 \in U_1} \left[\left(\frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)),\ \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \right) \cdot \mathbf{w} \right] \tag{4.78}$$

where $\mathbf{w}$ is a vector which depends on t, $\mathbf{x}^*(t)$ and $\mathbf{u}_2^*(t)$. Since $\mathbf{x}^*$ is the optimal trajectory, we know that the "best" direction $\mathbf{w}$ in the problem (4.78) is $\mathbf{w} = \frac{\mathrm{d}P}{\mathrm{d}t} = (1, \dot{\mathbf{x}}^*(t))$ (in the points t where $\dot{\mathbf{x}}^*(t)$ exists); using the dynamics we obtain

$$\begin{aligned} \mathbf{u}_1^*(t) &\in \arg\max_{\mathbf{u}_1 \in U_1} \left(\frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1, \mathbf{u}_2^*(t)) \right) \\ &= \arg\max_{\mathbf{u}_1 \in U_1} \left(\nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1, \mathbf{u}_2^*(t)) \right) \end{aligned} \tag{4.79}$$

Since the Player II wants to minimize, with similar arguments we obtain

$$\mathbf{u}_2^*(t) \in \arg\min_{\mathbf{u}_2 \in U_2} \left(\nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2) \right) \tag{4.80}$$

Now, by (4.77) and the dynamics, we have

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot \dot{\mathbf{x}}^*(t) \\ &= \frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2^*(t)) \\ \text{(by (4.79))} \quad &= \frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \max_{\mathbf{u}_1 \in U_1} \left(\nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1, \mathbf{u}_2^*(t)) \right) \\ \text{(by (4.80))} \quad &= \frac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + \min_{\mathbf{u}_2 \in U_2} \left(\nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2) \right). \end{aligned} \tag{4.81}$$

Let us conclude the proof using the Isaacs condition; for every fixed t, let us introduce the function

$$\widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2) = \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}_1, \mathbf{u}_2). \tag{4.82}$$

The previous equalities give that

$$\widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2^*(t)) = \max_{\mathbf{u}_1 \in U_1} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2^*(t)) = \min_{\mathbf{u}_2 \in U_2} \widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2). \tag{4.83}$$

If we show that

$$\widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2^*(t)) = \max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2) = \min_{\mathbf{u}_2 \in U_2} \max_{\mathbf{u}_1 \in U_1} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2) \tag{4.84}$$

we obtain $0 = \dfrac{\partial V}{\partial t}(t, \mathbf{x}^*(t)) + H_{DP}(t, \mathbf{x}^*(t), \nabla_{\mathbf{x}} V(t, \mathbf{x}^*(t)))$ and the proof is finished (note that in (4.84) the second equality is true by the Isaacs' condition). Hence let us suppose, in order to obtain a contradiction, that (4.84) is false: then

$$\begin{aligned}
\widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2^*(t)) \quad & =_{(4.83)} \max_{\mathbf{u}_1 \in U_1} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2^*(t)) \\
& >_{((4.84)\, false)} \min_{\mathbf{u}_2 \in U_2} \max_{\mathbf{u}_1 \in U_1} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2) \\
& =_{(4.75)} \max_{\mathbf{u}_1 \in U_1} \min_{\mathbf{u}_2 \in U_2} \widetilde{h}_t(\mathbf{u}_1, \mathbf{u}_2) \\
& \geq \min_{\mathbf{u}_2 \in U_2} \widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2) \\
& =_{(4.83)} \widetilde{h}_t(\mathbf{u}_1^*(t), \mathbf{u}_2^*(t)),
\end{aligned}$$

which clearly is impossible. Hence (4.84) holds: now using (4.81)–(4.84) we obtain (4.76). □

4.3.1 Examples

Example 4.3.1. *(see [2]) Let us consider the two-person zero-sum game*

$$\begin{cases}
\text{Player I: } \max_{u_1} J(u_1, u_2), \qquad \text{Player II: } \min_{u_2} J(u_1, u_2) \\
J(u_1, u_2) = \dfrac{1}{2} \displaystyle\int_0^2 (u_2^2 - u_1^2)\, \mathrm{d}t + \dfrac{1}{2} x(2)^2 \\
\dot{x} = \sqrt{2} u_2 - u_1 \\
x(0) = x_0
\end{cases}$$

First, it is easy to show that the Isaacs' condition holds:

$$H_{DP}^-(t, x, \lambda) = \max_{u_1 \in \mathbb{R}} \left(-\frac{1}{2} u_1^2 - \lambda u_1 \right) + \min_{u_2 \in \mathbb{R}} \left(\frac{1}{2} u_2^2 + \sqrt{2} \lambda u_2 \right) = H_{DP}^+(t, x, \lambda). \tag{4.85}$$

Hence we can define the Hamiltonian of Dynamic Programming H_{DP} : in order to do that, note that

$$u_1^* = -\lambda^*, \qquad u_2^* = -\sqrt{2}\lambda^* \tag{4.86}$$

realize the max and min in (4.85) respectively and hence $H_{DP}(t, x, \lambda) = -\frac{1}{2}\lambda^2$. Hence there exists the value function V that solves the Isaacs' equation

$$\begin{cases}
\dfrac{\partial V}{\partial t}(t, x) - \dfrac{1}{2} \left(\dfrac{\partial V}{\partial x}(t, x) \right)^2 = 0 & \text{for } (t, x) \in [0, 2] \times \mathbb{R} \\
V(2, x) = \dfrac{1}{2} x^2 & \text{for } x \in \mathbb{R}
\end{cases} \tag{4.87}$$

Since we are considering an Affine–Quadratic problem, we looking for a value function of the type (2.18), i.e. $V(t, x) = \frac{1}{2} Z(t) x^2 + W(t) x + Y(t)$: replacing such V in (4.87) we obtain

$$\dot{Z} x^2 + 2\dot{W} x + 2\dot{Y} - (Zx + W)^2 = 0, \quad \forall (t, x) \in [0, 2] \times \mathbb{R}$$

$$\Rightarrow \dot{Z} = Z^2 \tag{4.88}$$

$$\dot{W} = ZW \tag{4.89}$$

$$2\dot{Y} = W^2 \tag{4.90}$$

$$V(2, x) = \frac{1}{2} Z(2) x^2 + W(2) x + Y(2) = \frac{1}{2} x^2, \quad \forall x \in \mathbb{R}$$

$$\Rightarrow Z(2) = 1, \quad W(2) = Y(2) = 0 \tag{4.91}$$

Easy computations give: by (4.88) and (4.91) $Z(t) = \frac{1}{3-t}$*; by (4.89) and (4.91),* $W(t) = 0$*; finally, by (4.90) and (4.91),* $Y(t) = 0$*. Hence we obtain the value function*

$$V(t,x) = \frac{1}{2(3-t)}x^2.$$

Now, relation (4.72) gives $\lambda^*(t) = \frac{1}{3-t}x^*(t)$*, where* x^* *is the optimal trajectory; (4.86) gives*

$$u_1^*(t) = -\frac{1}{3-t}x^*(t), \qquad u_2^*(t) = -\frac{\sqrt{2}}{3-t}x^*(t). \tag{4.92}$$

Hence we are in the position to find a feedback strategy. Using this expression for the control (u_1, u_2)*, the dynamics gives*

$$\dot{x}^* = -\frac{x^*}{3-t};$$

together with the initial condition $x(0) = x_0$*, we obtain* $x^* = x_0\frac{3-t}{3}$*. This implies that* (u_1^*, u_2^*) *in (4.92) is a feedback Nash equilibrium.*

△

4.4 Pursuit-evasion games

Let

$$\mathcal{T} = \mathbb{R}^+ \times \mathcal{T}_0 \subset \mathbb{R}^+ \times \mathbb{R}^n$$

be a target set, with $\mathcal{T}_0$ closed. Let as denote by $\mathcal{G} \subset \mathbb{R}^+ \times \mathbb{R}^n$ the game set, i.e. the set where the trajectories lie, $(t, \mathbf{x}(t)) \in \mathcal{G}$.

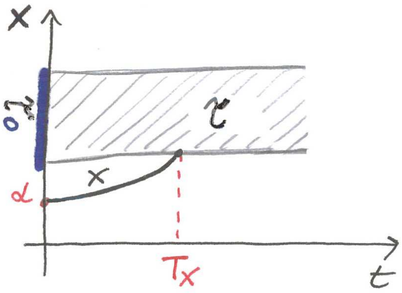
We investigate a situation in which the first player tries to maintain the state of the system as long as possible outside to a target set $\mathcal{T}$ while the second player aims reaching $\mathcal{T}$ as soon as possible. For this reasons, in all this section, the first player, who chooses $\mathbf{u}_1$, is called Pursuer, which we shall abbreviate buy P; the second player, who chooses $\mathbf{u}_2$, is called Evader, which we shall abbreviate buy E.
We consider an autonomous problem, i.e. a situation where f, g and ψ do not depend directly on t. For every $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t, \mathbf{x}^*(t))$, with the corresponding $\mathbf{x}$ via

$$\dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu}_1(t,\mathbf{x}), \mathbf{u}_2(t,\mathbf{x})), \qquad \mathbf{x}(0) = \boldsymbol{\alpha}$$

such that $(0, \boldsymbol{\alpha}) \in \mathcal{G} \setminus \mathcal{T}$, the exit time in (4.74) is

$$T_{\mathbf{x}} = \inf\{t \geq 0 : \mathbf{x}(t) \in \mathcal{T}_0\}; \tag{4.93}$$

if the initial data on the trajectory $\mathbf{x}$ is $\mathbf{x}(\tau) = \boldsymbol{\xi}$, with $(\tau, \boldsymbol{\xi}) \in \mathcal{G} \setminus \mathcal{T}$, then the definition of its exit time is $T_{\mathbf{x}} = \inf\{t \geq \tau : \mathbf{x}(t) \in \mathcal{T}_0\}$.



We are interested in the game

$$\begin{cases} \textit{Pursuer}: \min\limits_{\mathbf{u}_1} J(\mathbf{u}_1, \mathbf{u}_2), \qquad \textit{Evader}: \max\limits_{\mathbf{u}_2} J(\mathbf{u}_1, \mathbf{u}_2) \\ \quad \mathbf{u}_1(t) \in U_1, \qquad\qquad \mathbf{u}_2(t) \in U_2 \\ \quad J(\mathbf{u}_1, \mathbf{u}_2) = \int_0^{T_{\mathbf{x}}} f(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}t + \psi(\mathbf{x}(T_{\mathbf{x}})) \\ \quad \dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \\ \quad \mathbf{x}(0) = \boldsymbol{\alpha}, \qquad (0, \boldsymbol{\alpha}) \in \mathcal{G} \setminus \mathcal{T} \\ \quad (T_{\mathbf{x}}, \mathbf{x}(T_{\mathbf{x}})) \in \partial\mathcal{T} \end{cases} \tag{4.94}$$

where $T_{\mathbf{x}}$ is the exit time of the trajectory $\mathbf{x}$. Note that in the Pursuit-Evasion games the first Player (P) would like to have a min, while (E) wishes to have a max; this notation is in honor of Isaacs and it is exactly as in his book [10] (see page 201). Clearly, all the results of the previous sections hold with easy modifications.

Here we have that H_{DP}^- and H_{DP}^+ do not depend on t. For this type of problems we have the following properties:

Proposition 4.1. *Let us consider the game (4.94) with f, g and ψ continuous, and with $\mathcal{T}_0$ closed. Then*

i. the lower V^- value function does not depend explicitly on t in the game set $\mathcal{G}$, i.e.

$$V^-(t,\mathbf{x}) = V^-(\mathbf{x}), \qquad \forall (t,\mathbf{x}) \in \mathcal{G}; \tag{4.95}$$

ii. the game set $\mathcal{G}$ for the game (4.94) is

$$\mathcal{G} = \mathbb{R}^+ \times \mathcal{G}_0 \subset \mathbb{R}^+ \times \mathbb{R}^n;$$

iii. if the lower value function V^- is in $C^1(\mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0))$, then the lower Isaacs' equation (4.45) becomes

$$\begin{cases} H^-_{DP}(\mathbf{x}, \nabla V^-(\mathbf{x})) = 0 & \textit{for } \mathbf{x} \in \mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0) \\ V^-(\mathbf{x}) = \psi(\mathbf{x}) & \textit{for } \mathbf{x} \in \mathcal{T}_0 \end{cases}$$

Similar results in i. and iii. hold for V^+.

Proof. Let $(\tau,\boldsymbol{\xi}) \in \mathcal{T}$; clearly $V^-(\tau,\boldsymbol{\xi}) = \psi(\boldsymbol{\xi})$. Now, let $(\tau,\boldsymbol{\xi}) \in \mathcal{G} \setminus \mathcal{T}$: by definition[3],

$$V^-(\tau,\boldsymbol{\xi}) = \inf_{\boldsymbol{\nu}_1} \sup_{\boldsymbol{\nu}_2} \int_\tau^{T_\mathbf{x}} f(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,\mathrm{d}t + \psi(\mathbf{x}(T_\mathbf{x}))$$

where in the previous line $(\mathbf{u}_1,\mathbf{u}_2) \in \mathcal{A}_{FB}$, with $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(t,\mathbf{x}(t))$, and the corresponding $\mathbf{x}$ solution of of the ODE

$$\begin{cases} \dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu}_1(t,\mathbf{x}), \boldsymbol{\nu}_2(t,\mathbf{x})) & \text{in } [\tau, T_\mathbf{x}] \\ \mathbf{x}(\tau) = \boldsymbol{\xi} \end{cases}$$

For every such $(\mathbf{u}_1,\mathbf{u}_2) \in \mathcal{A}_{FB}$, let us consider $\widetilde{\boldsymbol{\nu}}_i(s,\mathbf{x}) = \boldsymbol{\nu}_i(\tau+s,\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and $s \geq 0$. Since g does not depends on t, the unique solution $\widetilde{\mathbf{x}}$ of the ODE

$$\begin{cases} \dot{\mathbf{x}} = g(\mathbf{x}, \widetilde{\boldsymbol{\nu}}_1(t,\mathbf{x}), \widetilde{\boldsymbol{\nu}}_2(t,\mathbf{x})) \\ \mathbf{x}(0) = \boldsymbol{\xi} \end{cases}$$

is $\widetilde{\mathbf{x}}(s) = \mathbf{x}(\tau+s)$. Hence we consider $(\widetilde{\mathbf{u}}_1,\widetilde{\mathbf{u}}_2) \in \mathcal{A}_{FB}$, with $\widetilde{\mathbf{u}}_i(t) = \widetilde{\boldsymbol{\nu}}_i(t,\widetilde{\mathbf{x}}(t))$. Since $\mathcal{T} = \mathbb{R}^+ \times \mathcal{T}_0$, it is easy to see that

$$T_{\widetilde{\mathbf{x}}} = T_\mathbf{x} - \tau.$$

Clearly we obtain

$$\int_0^{T_{\widetilde{\mathbf{x}}}} f(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}}_1,\widetilde{\mathbf{u}}_2)\,\mathrm{d}t + \psi(\widetilde{\mathbf{x}}(T_{\widetilde{\mathbf{x}}})) = \int_\tau^{T_\mathbf{x}} f(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,\mathrm{d}t + \psi(\mathbf{x}(T_\mathbf{x})),$$

for every $(\mathbf{u}_1,\mathbf{u}_2)$ as before. This implies $V^-(\tau,\boldsymbol{\xi}) = V^-(0,\boldsymbol{\xi})$, for every $\tau > 0$. This proves *i.* and the arguments of this proof imply easily *ii.*

The assumption $V^- \in C^1(\mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0))$ and relation (4.95) give $\frac{\partial V^-}{\partial t}(t,\mathbf{x}) = 0$. Now, since the problem is autonomous, H^-_{DP} does not depend explicitly by t and we obtain, by the lower Isaacs' equation (4.45),

$$\frac{\partial V^-}{\partial t}(t,\mathbf{x}) + H^-_{DP}(t,\mathbf{x},\nabla_\mathbf{x} V^-(t,\mathbf{x})) = H^-_{DP}(\mathbf{x},\nabla V^-(\mathbf{x})) = 0.$$

Now, let us consider a initial data $(\tau,\boldsymbol{\xi})$ for our trajectory $\mathbf{x}$ in $\mathcal{T}$; it is clear that $T_\mathbf{x} = \tau$ and hence

$$V^-(\tau,\boldsymbol{\xi}) = \inf_{\boldsymbol{\nu}_1} \sup_{\boldsymbol{\nu}_2} \int_\tau^{T_\mathbf{x}} f(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2)\,\mathrm{d}t + \psi(\mathbf{x}(T_\mathbf{x})) = \psi(\boldsymbol{\xi}).$$

Taking into account that V^- does not depend explicitly by t, we have *iii.*. □

The next result is an easy consequence of the previous proposition.

[3] We recall that the First Player minimizes and the second Player maximizes, hence in the definition of V^+ and V^- we have to change 1 with 2 and viceversa.

Remark 4.4. *Let us consider the game (4.94) with f, g and ψ continuous, and with $\mathcal{T}_0$ closed. If the Isaacs' condition is satisfied, i.e.*

$$\begin{aligned} H_{DP}(\mathbf{x}, \boldsymbol{\lambda}) &= \min_{\mathbf{v}_1 \in U_1} \max_{\mathbf{v}_2 \in U_2} \Big(f(\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2) + \boldsymbol{\lambda} \cdot g(\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2) \Big) \\ &= \max_{\mathbf{v}_2 \in U_2} \min_{\mathbf{v}_1 \in U_1} \Big(f(\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2) + \boldsymbol{\lambda} \cdot g(\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2) \Big). \end{aligned} \tag{4.96}$$

and the value function V is in $C^1(\mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0))$, then V does not depend explicitly on t. Moreover the Isaacs' equation (4.62) becomes

$$\begin{cases} H_{DP}(\mathbf{x}, \nabla V(\mathbf{x})) = 0 & \text{for } \mathbf{x} \in \mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0) \\ V(\mathbf{x}) = \psi(\mathbf{x}) & \text{for } \mathbf{x} \in \mathcal{T}_0 \end{cases} \tag{4.97}$$

Theorem 4.7 for our pursuit-evasion game has a new statement:

Theorem 4.9. *Let us consider the game (4.94) with f, g and ψ continuous, and with $\mathcal{T}_0$ closed. Let us suppose that the Isaacs' condition is satisfied. Let us suppose that there exists a continuous differentiable function $V : (\mathcal{G}_0 \setminus \mathcal{T}_0) \to \mathbb{R}$ such that the Isaacs' equation (4.97) holds.*
Let $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i^*(t) = \boldsymbol{\nu}_i^*(\mathbf{x}^*(t))$, with the corresponding trajectory $\mathbf{x}^*$ with $\mathbf{x}^*(0) = \boldsymbol{\alpha}$ and exit time $T_{\mathbf{x}^*}$, be such that $(\boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}))$ realizes, for every $\mathbf{x}$, the max-min *in the Isaacs equation, i.e.*

$$0 = f(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) + \nabla V(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = H_{DP}(\mathbf{x}, \nabla V(\mathbf{x})), \tag{4.98}$$

for every $\mathbf{x} \in \mathrm{int}(\mathcal{G}_0 \setminus \mathcal{T}_0)$. Then $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is a Nash equilibrium.

We note that in our pursuit-evasion games (4.94), the feedback strategies $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{A}_{FB}$ are such that $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(\mathbf{x}(t))$ with $\boldsymbol{\nu}_i$ which depends only on $\mathbf{x}$, i.e. we looking for stationary feedback strategies. Finally, along the optimal trajectory (4.98) becomes

$$0 = f(\mathbf{x}^*(t), \boldsymbol{\nu}_1^*(\mathbf{x}^*(t)), \boldsymbol{\nu}_2^*(\mathbf{x}^*(t))) + \nabla V(\mathbf{x}^*(t)) \cdot g(\mathbf{x}^*(t), \boldsymbol{\nu}_1^*(\mathbf{x}^*(t)), \boldsymbol{\nu}_2^*(\mathbf{x}^*(t))), \tag{4.99}$$

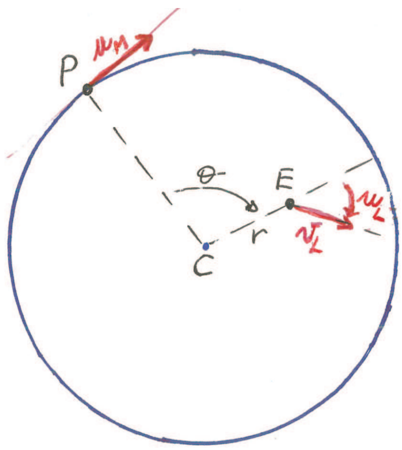
for every $t \in [0, T_{\mathbf{x}^*}]$.

4.4.1 The lady in the lake

The following games is in [2]. A lady (E=Evader) is swimming in a circular lake (of radius R) with a velocity $\overrightarrow{v_L}(t)$ such that $v_L(t) = v_L$ is constant and $v_L < 1$; she can change the direction in which she swims instantaneously. Hence the lady controls the direction of its velocity, i.e. she controls the angular velocity u_L with respect to the radius $\overline{CE}$, without any restriction.

A man (P=Pursuer) is not a swimmer and he wishes to intercept the lady when she reaches the shore; he is in the beach of the lake and can run along the perimeter with velocity $\overrightarrow{u_M}(t)$ which is tangent to the circumference. He also can change his direction instantaneously; hence the man controls the signed modulo u_M, i.e. $|u_M(t)| \le 1$, where $u_M > 0$ ($u_M < 0$) implies that the man runs clockwise (counter-clockwise) around the lake.

We assume that the lady and the man never get tired. E doesn't stay in the lake forever and she wishes to come out without being caught by the man; in the land, E can run faster than P. E's goal is to maximize the pay-off, which is the angular distance θ viewed from the center C of the lake, at the time E reaches to the store. P obviously wants minimize such pay-off.



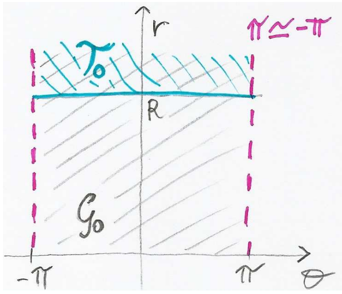
In order to describe the system, we introduce the angular distance $\theta = \theta(t)$, i.e. the angle between P and E with respect to the center C in a clockwise sense: we consider $-\pi \le \theta(t) \le \pi$ and the identification $\pi \simeq -\pi$. The E's distance with respect to the center of the lake is $r = r(t)$. The dynamics of the game (we left to the reader the details) is

$$\dot{\theta} = \frac{v_L \sin u_L}{r} - \frac{u_M}{R} \tag{4.100}$$
$$\dot{r} = v_L \cos u_L \tag{4.101}$$

The pay-off function is $|\theta(T)|$, where $T = T(\theta, r)$ is defined for every trajectory (θ, r), as in (4.93), by

$$T = \inf\{t \ge 0 : \ (\theta(t), r(t)) \in \mathcal{T}_0\},$$

where $\mathcal{T}$ is the target set of the game, $\mathcal{T} = \mathbb{R}^+ \times \mathcal{T}_0$ with $\mathcal{T}_0 = [-\pi, \pi] \times [R, \infty)$ (and $\pi \simeq -\pi$). We are in a pursuit-evasion game as in (4.94). Proposition 4.1 implies that the game set $\mathcal{G}$ for our problem is $\mathcal{G} = \mathbb{R}^+ \times \mathcal{G}_0$: let us note that for every (r_0, θ_0) with $r_0 \ge 0$ and $\theta_0 \in [-\pi, \pi]$ the pair $(u_M, u_L) = (0, 0)$ in the dynamics implies the trajectory $r(t) = r_0 + t v_L$ and $\theta(t) = \theta_0$ for every $t \ge 0$: it is clear that for some t the trajectory reaches in the target set: hence



$$\mathcal{G}_0 = [-\pi, \pi] \times [0, \infty),$$

with $\pi \simeq -\pi$.

Hence we have the pursuit-evasion game

$$\begin{cases} \textit{Man (P)}: \ \min\limits_{u_M} |\theta(T)| \qquad\qquad \textit{Lady (E)}: \ \max\limits_{u_L} |\theta(T)| \\ \quad |u_M| \le 1 \\ \qquad \dot{\theta} = \dfrac{v_L \sin u_L}{r} - \dfrac{u_M}{R} \\ \qquad \dot{r} = v_L \cos u_L \\ \qquad r(0) = 0, \ r(T) = R \end{cases}$$

Let us looking for a Nash equilibrium (u_M^*, u_L^*) for this game in the family of (stationary) feedback strategies, where

$$u_M^*(t) = \nu_M^*(\theta^*(t), r^*(t)), \quad u_L^*(t) = \nu_L^*(\theta^*(t), r^*(t))$$

and the associated trajectory is (θ^*, r^*). In order to do that, we apply Theorem 4.9. The Hamiltonian $H = H(\theta, r, u_M, u_L, \lambda_1, \lambda_2)$ is

$$H = \lambda_1 \left(\frac{v_L \sin u_L}{r} - \frac{u_M}{R} \right) + \lambda_2 v_L \cos u_L.$$

For the upper and lower Hamiltonians of Dynamic Programming we have

$$\begin{aligned} H_{DP}^+(\theta, r, \lambda_1, \lambda_2) &= \min_{|u_M| \le 1} \max_{u_L} H(\theta, r, u_M, u_L, \lambda_1, \lambda_2) \\ &= \min_{|u_M| \le 1} \left\{ -\frac{\lambda_1}{R} u_M \right\} + \max_{u_L} \left\{ \frac{v_L \sin u_L}{r} \lambda_1 + \lambda_2 v_L \cos u_L \right\} \\ &= \max_{u_L} \min_{|u_M| \le 1} H(\theta, r, u_M, u_L, \lambda_1, \lambda_2) \\ &= H_{DP}^-(\theta, r, \lambda_1, \lambda_2) \end{aligned}$$

Hence the Isaacs' condition is satisfied and, by Proposition 4.1, we are in the position to looking for a value function V that does not depend on t, i.e. $V = V(\theta, r)$; let us notice that the existence of such V is not guaranteed since the dynamics is not bounded and Lipschitz w.r.t. r (see Theorem 4.6). The Isaacs' equation (4.97) is

$$\begin{cases} \min\limits_{|u_M|\le 1} \max\limits_{u_L} \left[\left(\frac{v_L \sin u_L}{r} - \frac{u_M}{R} \right) \frac{\partial V}{\partial \theta}(\theta, r) + v_L \cos u_L \frac{\partial V}{\partial r}(\theta, r) \right] = 0 & \text{for } (\theta, r) \in [-\pi, \pi] \times (0, R) \\ V(\theta, R) = |\theta| & \text{for } \theta \in [-\pi, \pi] \end{cases}$$

where the order of the min and the max is irrelevant. If (u_M^*, u_L^*) is our Nash equilibrium with associated trajectory (θ^*, r^*), let us set

$$V_\theta^*(t) = \frac{\partial V}{\partial \theta}(\theta^*(t), r^*(t)), \qquad V_r^*(t) = \frac{\partial V}{\partial r}(\theta^*(t), r^*(t)).$$

Let us reorganize the Isaacs' equation taking into account that (ν_M^*, ν_L^*) realizes the min and the max; in particular along the optimal path (θ^*, r^*), as in (4.99), we have

$$\begin{aligned} 0 &= \min_{|u_M|\le 1} \left(-\frac{u_M}{R} V_\theta^*(t) \right) + v_L \max_{u_L} \left(\frac{\sin u_L}{r^*(t)} V_\theta^*(t) + \cos u_L V_r^*(t) \right) \\ &= -\frac{u_M^*(t)}{R} V_\theta^*(t) + v_L \left(\frac{\sin(u_L^*(t))}{r^*(t)} V_\theta^*(t) + \cos(u_L^*(t)) V_r^*(t) \right) \end{aligned} \tag{4.102}$$

for every $t \in [0, T]$.

We can apply the variational approach for the open–loop representation of the feedback Nash equilibrium (u_M^*, u_L^*) (see Theorem 4.2). Taking into account that (see (4.72)) $\lambda_1^* = V_\theta^*$ and $\lambda_2^* = V_r^*$ along the optimal trajectory $(\theta^*(t), r^*(t))$ and for the time t such that V is sufficiently regular[4] in the point $(\theta^*(t), r^*(t))$, the adjoint equation gives

$$\dot\lambda_1^* = -\frac{\partial H}{\partial \theta} = 0 \qquad \Rightarrow \qquad V_\theta^* = k \tag{4.103}$$

$$\dot\lambda_2^* = -\frac{\partial H}{\partial r} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t} V_r^* = V_\theta^* \frac{v_L \sin u_L^*}{(r^*)^2} = k \frac{v_L \sin u_L^*}{(r^*)^2} \tag{4.104}$$

where k is a constant.

• First, let us suppose that for some time t on the optimal trajectory we have $V_\theta^*(t) \neq 0$: (4.102) implies that the optimal control u_M^* is given by

$$u_M^*(t) = \operatorname{sgn}(V_\theta^*(t)). \tag{4.105}$$

Consequently, (4.102) is

$$\max_{u_L} \left\{ \sin u_L \frac{V_\theta^*(t)}{r^*(t)} + \cos u_L V_r^*(t) \right\} = \frac{|V_\theta^*(t)|}{v_L R} \tag{4.106}$$

This implies that the vector $(\cos(u_L^*(t)), \sin(u_L^*(t)))$ has the same direction of the vector $\left(V_r^*(t), \frac{V_\theta^*(t)}{r^*(t)} \right)$ and hence

$$\left\| \left(V_r^*(t), \frac{V_\theta^*(t)}{r^*(t)} \right) \right\| = \frac{|V_\theta^*(t)|}{v_L R}. \tag{4.107}$$

An explicit calculation of the modulo gives

$$\frac{(V_\theta^*(t))^2}{(r^*(t))^2} \le (V_r^*(t))^2 + \frac{(V_\theta^*(t))^2}{(r^*(t))^2} = \frac{(V_\theta^*(t))^2}{v_L^2 R^2};$$

this last relation is true only in the case $r^*(t) \ge v_L R$.

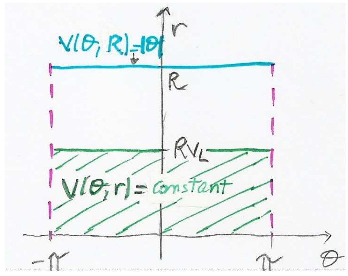
[4]Relation (4.103) seems to give that $t \mapsto \lambda_1^*(t) = V_\theta^*(t)$ is a constant in $[0, T]$: as we will see, this is not true since for $t = T_0$ (see below for the definition of T_0) V is not regular.

• Let's study the situation $r^*(t) < v_L R$: in this case, the previous calculations imply $V_\theta^*(t) = 0$: hence by (4.104) we obtain $V_r^*(t) = k_1$ constant and (4.102) becomes

$$\max_{u_L} \{k_1 \cos u_L\} = 0.$$

Such relation gives $k_1 = 0 = V_r^*(t)$. Hence on the optimal trajectory $(\theta^*(t), r^*(t))$ inside the circumference of radius $v_L R$ we have

$$\nabla V(\theta^*(t), r^*(t)) = (V_\theta^*(t), V_r^*(t)) = (0,0) :$$



the value function doesn't change if E modifies her position, i.e. V is constant. In this situation the Lady can achieve a large angular velocity w_L (with respect to the center C) than the angular velocity w_M of the Man P, and therefore she can always move herself into the position $\theta^*(t) = \pi$, i.e. to a position diametrically opposite from P. In fact we note that

$$w_L = \frac{v_L \sin u_L^*}{r^*}, \qquad w_M = \frac{u_M^*}{R} \tag{4.108}$$

and for $|w_L| \geq |w_M|$ implies

$$\frac{v_L}{r^*} \geq \left| \frac{v_L \sin u_L^*}{r^*} \right| \geq \left| \frac{u_M^*}{R} \right| = \frac{1}{R} :$$

hence $r^*(t) \leq v_L R$.

Clearly circumference of radius $v_L R$, for every strategy-decision ν_M^* of the Man, the Lady can consider a strategy-decision ν_L^* in order to stay in a situation where $\theta^*(t) = \pm\pi$; hence $\dot{\theta}^* = 0$ and $w_L = w_M$. Relations (4.108) give the optimal control

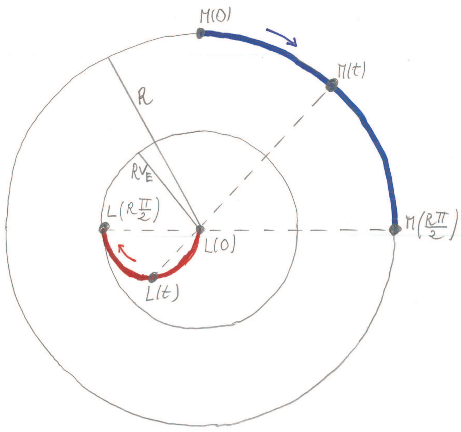
$$(u_M^*(t), u_L^*(t)) = \left(u_M^*(t), \arcsin \frac{u_M^*(t) r^*(t)}{R v_L} \right)$$

and the optimal trajectory is, again by the dynamics,

$$(|\theta^*(t)|, r^*(t)) = \left(\pi, R v_L \frac{\sin u_L^*(t)}{u_M^*(t)} \right). \tag{4.109}$$

An example of the situation $r(t) < v_L R$:
Let us suppose that the Man starts at the North pole of the circumference and runs in the clockwise sense with maximal velocity, i.e. $u_M^* = 1$. Let us denote by $M(t)$ the position of the Man at the time t, we have (see the blue curve in the picture)

$$M(t) = R\left(\sin \frac{t}{R}, \cos \frac{t}{R} \right).$$



Note that $|\dot{M}| = 1$. Let us denote by $L(t)$ the position of the Lady at time t: (4.109) gives

$$r^* = R v_L \sin u_L^* \quad \Rightarrow \quad \dot{r^*} = R v_L \dot{u}_L^* \cos u_L^*.$$

Since the dynamics is $\dot{r^*} = v_L \cos u_L^*$

we obtain $R\dot{u}_L^* = 1$, i.e. $u_L^*(t) = t/R + a$ with a constant. The initial condition $r^*(0) = 0$ with (4.109) give $a = 0$. Hence we have the strategy for the Lady

$$u_L^* = \frac{t}{R}.$$

By using (4.109), we obtain (see the red curve in the picture)

$$L(t) = -R v_L \sin \frac{t}{R} \left(\sin \frac{t}{R}, \cos \frac{t}{R} \right).$$

It is easy to verify that the modulo of the velocity of the Lady is exactly v_L.

• Since the lady doesn't stay in the lake forever, let us define T_0 as

$$T_0 = \max\{t \geq 0;\ r^*(t) = v_L R\},$$

and let us study the situation $r^*(t) \geq v_L R$. The previous argument imply that

$$|\theta^*(T_0)| = \pi, \qquad r^*(T_0) = RV_L. \tag{4.110}$$

Taking into account that, by the final condition on the value function $V(\theta, R) = |\theta|$, we have $V_\theta^*(T) = \frac{\partial V}{\partial \theta}(\theta^*(T), r^*(T)) = \mathrm{sgn}(\theta^*(T))$: equation (4.103) gives

$$V_\theta^*(t) = V_\theta^*(T), \qquad t \in [T_0, T].$$

Hence, by (4.105),

$$u_M^*(t) = \mathrm{sgn}(\theta^*(T)), \qquad t \in [T_0, T]. \tag{4.111}$$

Relation (4.106) gives

$$(\cos(u_L^*(t)), \sin(u_L^*(t))) = \frac{v_L R}{|V_\theta^*(t)|}\left(V_r^*(t), \frac{V_\theta^*(t)}{r^*(t)}\right).$$

Hence

$$\sin u_L^*(t) = \frac{v_L R}{r^*(t)}\mathrm{sgn}(\theta^*(T)). \tag{4.112}$$

The optimal control here is

$$(u_M^*(t), u_L^*(t)) = \left(\mathrm{sgn}(\theta^*(T)), \arcsin\frac{v_L R\mathrm{sgn}(\theta^*(T))}{r^*(t)}\right), \qquad t \in [T_0, T].$$

• Let us discuss the possibility of the Lady and the Man obtain their respectively pay off. In $r^*(t) \geq v_L R$, the dynamics (4.100) and the optimal control (u_M^*, u_L^*) in (4.111) and (4.112) give

$$\dot{\theta}^* = \frac{v_L \sin u_L^*}{r^*} - \frac{u_M^*}{R} = \mathrm{sgn}(\theta^*(T))\frac{v_L^2 R^2 - (r^*)^2}{R(r^*)^2}. \tag{4.113}$$

Now, taking into account the dynamics (4.101) and the optimal control u_L^* in (4.112), we have

$$\dot{r}^* = v_L \cos u_L^* = \frac{v_L}{r^*}\sqrt{(r^*)^2 - v_L^2 R^2};$$

we consider only the case $\cos u_L^* > 0$ since the Lady cannot stay in the lake forever (see the dynamics). Hence (4.113) becomes

$$\begin{aligned} \dot{\theta}^* &= -\frac{\mathrm{sgn}(\theta^*(T))}{R}\frac{\sqrt{(r^*)^2 - v_L^2 R^2}}{r^*}\frac{\sqrt{(r^*)^2 - v_L^2 R^2}}{r^*} \\ &= -\frac{\mathrm{sgn}(\theta^*(T))}{v_L R}\frac{\sqrt{(r^*)^2 - v_L^2 R^2}}{r^*}\dot{r}^*. \end{aligned}$$

Taking int account (4.110), let us put $\theta^*(T_0) = \pi\mathrm{sgn}(\theta^*(T))$. Hence the last equality and (4.110) imply

$$\begin{aligned} &\int_{\theta^*(T_0)}^{\theta^*(T)} \mathrm{d}\theta = \int_{r^*(T_0)}^{r^*(T)} -\frac{\mathrm{sgn}(\theta^*(T))}{v_L R}\frac{\sqrt{r^2 - v_L^2 R^2}}{r}\,\mathrm{d}r \\ \Rightarrow\quad &(\theta^*(T) - \theta^*(T_0))\mathrm{sgn}(\theta^*(T)) = -\frac{1}{v_L R}\int_{v_L R}^{R}\frac{\sqrt{r^2 - v_L^2 R^2}}{r}\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad |\theta^*(T)| - \pi &= -\int_{v_L}^{1} \frac{\sqrt{1-s^2}}{s^2}\,\mathrm{d}s \qquad (\text{with } s = \frac{v_L R}{r}) \\
&= -\left[\left(-\frac{1}{s}\right)\sqrt{1-s^2}\right]_{v_L}^{1} + \int_{v_L}^{1} \frac{1}{\sqrt{1-s^2}}\,\mathrm{d}s \quad (\text{by part}) \\
&= -\frac{1}{v_L}\sqrt{1-v_L^2} + \Big(-\arccos s\Big]_{v_L}^{1} \\
&= -\frac{1}{v_L}\sqrt{1-v_L^2} + \arccos v_L.
\end{aligned}
$$

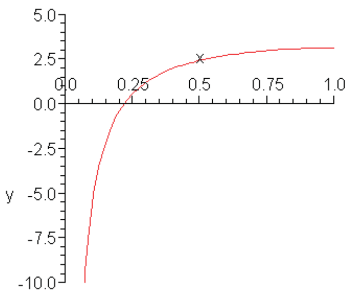
Clearly the previous calculation does not take into account the identification $\theta^*(\pi) = \theta^*(-\pi)$: we obtain

$$|\theta^*(T)| = \pi - \frac{1}{v_L}\sqrt{1-v_L^2} + \arccos v_L.$$

A plot of the function, for $x \in (0,1]$,

$$x \mapsto y(x) = \pi - \frac{1}{x}\sqrt{1-x^2} + \arccos x,$$

comes from $\lim_{x\to 0^+} y(x) = -\infty$, $y(1) = \pi$ and $y' = \frac{\sqrt{1-x^2}}{x^2} \geq 0$.



Hence there exists a value $v_L^* \in (0,1)$ such that for $v_L \in (v_L^*, 1)$ the Lady sure will not be caught by the Man.

4.5 Pursuit-evasion game of kind

We consider a situation similar to section 4.4, but now the functional J takes only a finite number of values: these type of pursuit-evasion games are called games of kind. The theory of these type of games is very wide and here we would like to give some ideas, in some particular situations (see for example [12] for more details).

Let us consider the problem in (4.94). Here we consider a closed target set $\mathcal{T} = \mathbb{R}^+ \times \mathcal{T}_0 \subset \mathcal{G}$, with $\mathrm{int}(\mathcal{T}_0) \neq \emptyset$, such that $\partial\mathcal{T}_0$ is a $(n-1)$-dimensional surface in C^1, i.e.

$$\partial\mathcal{T}_0 = \{\mathbf{x} \in \mathcal{G}_0 \subset \mathbb{R}^n : \; h(\mathbf{x}) = 0\},$$

where $h : \mathcal{G}_0 \to \mathbb{R}$ is a function in C^1. For every $\mathbf{x} \in \partial\mathcal{T}_0$, let us denote by $\mathbf{n}(\mathbf{x}) \in \mathbb{R}^n$ the outward normal of $\mathcal{T}_0$ at $\mathbf{x}$; clearly $\mathbf{n}(\mathbf{x}) \parallel \nabla h(\mathbf{x})$.
In this game, the Evader (E) tries to prevent the state–trajectory from reaching into the interior of $\mathcal{T}_0$, whereas the Pursuer (P) seeks the opposite. We assign numerical values to the outcomes J in (4.94) for the trajectory $\mathbf{x}$ associated to the strategy $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{A}_{FB}$, where $\mathbf{u}_i(t) = \boldsymbol{\nu}_i(\mathbf{x}(t))$:

- -1 for termination of the game or capture, i.e. the trajectory $\mathbf{x}$ arrives in $\mathrm{int}(\mathcal{T}_0)$;
- $+1$ for no termination of the game or escape, i.e. the trajectory $\mathbf{x}$ never arrives in $\mathrm{int}(\mathcal{T}_0)$.

Hence we are interested in the game

$$
\begin{cases}
\textit{Pursuer}: \min_{\mathbf{u}_1} J(\mathbf{u}_1,\mathbf{u}_2), \qquad \textit{Evader}: \max_{\mathbf{u}_2} J(\mathbf{u}_1,\mathbf{u}_2) \\
\qquad \mathbf{u}_1 \in U_1 \qquad\qquad\qquad\qquad \mathbf{u}_2 \in U_2 \\
\qquad J(\mathbf{u}_1,\mathbf{u}_2) = \begin{cases} -1 & \text{if } \exists\, t \geq 0 \text{ s.t. } \; \mathbf{x}(t) \in \mathrm{int}(\mathcal{T}_0) \\ +1 & \text{otherwise} \end{cases} \\
\qquad \dot{\mathbf{x}} = g(\mathbf{x},\mathbf{u}_1,\mathbf{u}_2) \\
\qquad \mathbf{x}(0) = \boldsymbol{\alpha}
\end{cases}
\tag{4.114}
$$

with g continuous function and $(0, \boldsymbol{\alpha}) \in \mathcal{G}$ fixed. For our game the Hamiltonian is

$$H(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2). \tag{4.115}$$

Let us suppose for this problem that the Isaacs' condition is satisfied.

It is clear that in this problem the regularity assumption in *1.* does not hold and, despite the Isaacs' assumption holds, it is not possible to guarantee that there exists the value function V. However we can define V^- and V^+ and, exactly as in *i.–ii.* of Proposition 4.1 we have

Remark 4.5. *For the game (4.114) we have that:*

i. the lower value functions V^- and the upper value function V^+ do not depend on t;

ii. the game set is $\mathcal{G} = \mathbb{R}^+ \times \mathcal{G}_0$.

Since the "solution" of the game does not depend on the time, the most interesting question is to study which initial points of $\mathcal{G}_0$ lead to a termination of the game and which are not. In order to do that, let us classify the points of the set $\mathcal{G}_0$:

States of capture, states of escape in $\mathcal{G}_0 \setminus \mathrm{int}(\mathcal{T}_0)$.

Definition 4.9. *Let us consider the game (4.114). Let $\mathbf{x}_0 \in \mathcal{G}_0 \setminus \mathrm{int}(\mathcal{T}_0)$. We say that*

- $\mathbf{x}_0$ *is a state of termination (or to capture) if the first Player (P) has a strategy–control $\overline{\mathbf{u}}_1$, i.e. a decision rule $\overline{\boldsymbol{\nu}}_1 = \overline{\boldsymbol{\nu}_1}(\mathbf{x})$, such that for every acts–strategy $\mathbf{u}_2$ of the second Player (E), i.e. for every decision rule $\boldsymbol{\nu}_2 = \boldsymbol{\nu}_2(\mathbf{x})$, the trajectory[5] can be steered to the interior of the target set: we denote by $\mathcal{C}_{ap}$ the set of all the states of capture;*
- $\mathbf{x}_0$ *is a state of no–termination (or to escape) if (E) has a strategy-control such that for every acts–strategy of (P) the trajectory can be steered outside to int$(\mathcal{T}_0)$ forever: we denote by $\mathcal{E}_{sc}$ the set of all the states of escape.*

Let us prove the following:

Remark 4.6. *If is $\mathbf{x} \in \mathcal{C}_{ap}$, then $V^-(\mathbf{x}) = V^+(\mathbf{x}) = -1$. If is $\mathbf{x} \in \mathcal{E}_{sc}$, then $V^-(\mathbf{x}) = V^+(\mathbf{x}) = +1$.*

Proof. Let $\boldsymbol{\xi}$ be a state of capture. Then there exists a control $\widetilde{\mathbf{u}}_1$ for the first Player such that for every $\mathbf{u}_2$ such that $(\widetilde{\mathbf{u}}_1, \mathbf{u}_2) \in \mathcal{A}_{FB}$, with trajectory $\mathbf{x}$ which starts from $\boldsymbol{\xi}$, we have $J(\widetilde{\mathbf{u}}_1, \mathbf{u}_2) = -1$. This implies that

$$J(\widetilde{\mathbf{u}}_1, \mathbf{u}_2) = -1, \qquad \forall (\widetilde{\mathbf{u}}_1, \mathbf{u}_2) \in \mathcal{A}_{FB}.$$

Hence, by (4.16) and by definition[6],

$$-1 \le V^-(\boldsymbol{\xi}) \le V^+(\boldsymbol{\xi}) = \sup_{\boldsymbol{\nu}_2} \inf_{\boldsymbol{\nu}_1} J(\mathbf{u}_1, \mathbf{u}_2) \le \sup_{\boldsymbol{\nu}_2} J(\widetilde{\mathbf{u}}_1, \boldsymbol{\nu}_2) = -1$$

Now, let $\boldsymbol{\xi}$ be a state of escape. Then there exists a control $\widetilde{\mathbf{u}}_2$ such that for every $\mathbf{u}_1$ such that $(\mathbf{u}_1, \widetilde{\mathbf{u}}_2) \in \mathcal{A}_{FB}$, with trajectory $\mathbf{x}$ which starts from $\boldsymbol{\xi}$, we have $J(\mathbf{u}_1, \widetilde{\mathbf{u}}_2) = +1$. This implies that

$$J(\mathbf{u}_1, \widetilde{\mathbf{u}}_2) = 1, \qquad \forall (\mathbf{u}_1, \widetilde{\mathbf{u}}_2) \in \mathcal{A}_{FB}.$$

[5] The trajectory, in this situation, is the solution $\mathbf{x}$ of the ODE

$$\begin{cases} \dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \overline{\boldsymbol{\nu}}_1(\mathbf{x}(t)), \boldsymbol{\nu}_2(\mathbf{x}(t))) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

and hence $\overline{\mathbf{u}}_1(t) = \overline{\boldsymbol{\nu}}_1(\mathbf{x}(t))$, $\mathbf{u}_2(t) = \boldsymbol{\nu}_2(\mathbf{x}(t))$.

[6] We recall that the First Player minimizes and the second Player maximizes, hence in the definition of V^+ and V^- we have to change 1 with 2 and viceversa.

Hence, by (4.16) and by definition,

$$1 \geq V^+(\boldsymbol{\xi}) \geq V^-(\boldsymbol{\xi}) = \inf_{\boldsymbol{\nu}_1} \sup_{\boldsymbol{\nu}_2} J(\boldsymbol{\nu}_1, \mathbf{u}_1) \geq \inf_{\boldsymbol{\nu}_1} J(\boldsymbol{\nu}_1, \widetilde{\mathbf{u}}_2) = 1.$$

□

Now we are in the position to define, with an abuse of language, a sort of value function

$$V : \mathcal{C}_{ap} \cup \mathcal{E}_{sc} \to \{-1, +1\}.$$

The most interesting situation, of course, is the one in which $\mathcal{G}_0$ contains both capture and escape states. It is clear that the value function V is discontinuous and the theory of the previous sections is not useful.

Usable part in $\partial\mathcal{T}_0$.

At this point a natural question is to investigate which points of the boundary of the target set are candidate to be a state of termination for the game. We have the following:

Definition 4.10. *Let us consider the game (4.114) and let us assume that the Isaacs' condition is satisfied. We define the usable part, that we will denote by $\mathcal{UP}$, as*

$$\mathcal{UP} = \left\{ \mathbf{x} \in \partial\mathcal{T}_0 : \min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{n}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{n}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \leq 0 \right\}. \tag{4.116}$$

The points of the usable part are candidate to be termination for the game. If strict inequality holds in (4.116) for some $\mathbf{x}$, then it is a state of termination, i.e. $\mathbf{x} \in \mathcal{C}_{ap}$, and it penetrates in $\mathcal{T}_0$; to be clear, if for some controls $(\mathbf{u}_1, \mathbf{u}_2)$ and for some time t, the trajectory $\mathbf{x}$ associated to such controls arrives at time t in a point of this type, then $\dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \mathbf{u}_1(t), \mathbf{u}_2(t))$ gives the direction of the trajectory in the point $\mathbf{x}(t)$; since $\mathbf{n}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) < 0$, then the trajectory enters in $\mathrm{int}(\mathcal{T}_0)$.
The points $\mathbf{x}$ for which equality holds in (4.116) may be only touching points.

The barrier and its construction.

We now are in the position to introduce the most important "object" that allows us to study our problem: the barrier. It is the set that separates the state of capture to the state of escape:

Definition 4.11. *We define the barrier $\mathcal{B}_{ar}$ the set in $\mathcal{G}_0 \setminus \mathrm{int}(\mathcal{T}_0)$ by*

$$\mathcal{B}_{ar} = \partial\mathcal{C}_{ap} \cap \partial\mathcal{E}_{sc}.$$

The boundary of the usable part, that we will denote by $\mathcal{BUP}$, is the set

$$\mathcal{BUP} = \mathcal{UP} \cap \mathcal{B}_{ar}.$$

It is clear that the barrier can be a very irregular set. Let us introduce the following two assumptions:
Assumption. *Let us assume that $\mathcal{B}_{ar}$ is non empty and*

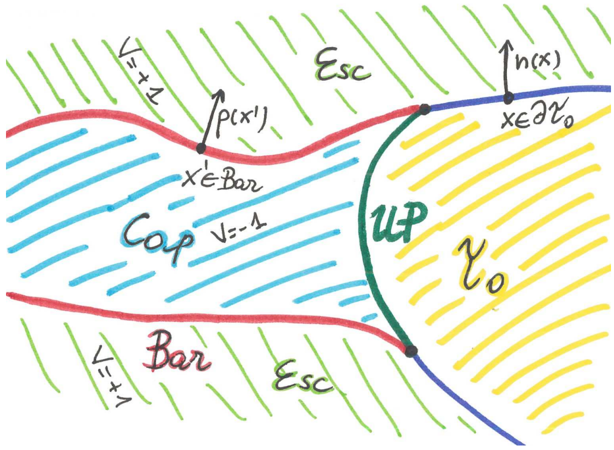
- (smoothness) *the barrier is a C^2 surface, i.e. $\mathcal{B}_{ar} = \{\mathbf{x} : b(\mathbf{x}) = 0\}$, with $b \in C^2$ and such that*

$$\mathbf{x} \in \mathcal{B}_{ar} \quad \Longrightarrow \quad \mathbf{x} \notin \mathcal{C}_{ap} \cup \mathcal{E}_{sc};$$

- (naturality) *the curve from which the barrier starts is $\mathcal{BUP}$, i.e. the boundary of the usable part.*

For every point $\mathbf{x} \in \mathcal{B}_{ar}$, let us denote by $\mathbf{p}(\mathbf{x}) \in \mathbb{R}^n$ the outward normal of $\mathcal{C}_{ap}$ in the point $\mathbf{x}$ (and inward $\mathcal{E}_{sc}$): clearly $\mathbf{p}(\mathbf{x}) \parallel \nabla b(\mathbf{x})$. Without loss of generality we assume

$$\mathbf{p}(\mathbf{x}) = \nabla b(\mathbf{x}). \tag{4.117}$$



A natural barrier: for every point $\mathbf{x}_0$ *in* $\mathcal{BUP}$*, we have* $\mathbf{p}(\mathbf{x}_0) = \mathbf{n}(\mathbf{x}_0)$ *(as we will see in (4.129)).*

If we are able to construct the barrier, then we would also have found $\mathcal{C}_{ap}$ and $\mathcal{E}_{sc}$ and, as a result, we solve the game.

A first result in the direction to construct $\mathcal{B}_{ar}$ requires this definition:

Definition 4.12. *Let us consider the game (4.114) and let us assume that the Isaacs' condition is satisfied. Let* $A \subset \mathcal{G}_0$ *be a regular surface, i.e. with*

$$A = \{\mathbf{x} \in \mathcal{G}_0 \subset \mathbb{R}^n : \ a(\mathbf{x}) = 0\},$$

where $a : \mathcal{G}_0 \to \mathbb{R}$ *is a function in* C^1*, and such that* $\mathbf{a}(\mathbf{x}) \in \mathbb{R}^n$ *is the normal in the point* $\mathbf{x}$ *to* A *(clearly* $\mathbf{a}(\mathbf{x}) \parallel \nabla a(\mathbf{x})$*). We say that* A *is a semipermeable surface if*

$$\min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{a}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{a}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = 0.$$

The property of the barrier is the following:

Proposition 4.2. *Let us assume that, for the game (4.114), the Isaacs' condition is satisfied,* g *is a continuous function with continuous derivative with respect to* $\mathbf{x}$*, and the barrier* $\mathcal{B}_{ar}$ *is non empty and smooth. Then the barrier is a semipermeable surface, i.e. for every* $\mathbf{x} \in \mathcal{B}_{ar}$ *we have*

$$\min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = 0. \tag{4.118}$$

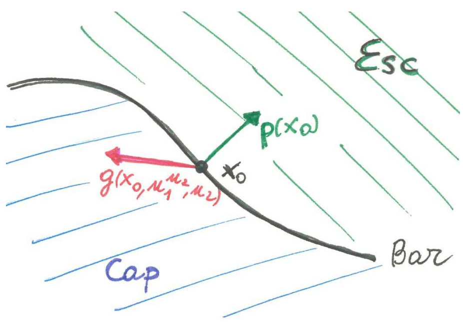
Proof. Let us consider $\mathbf{x}_0 \in \mathcal{B}_{ar}$; let β be the value of the previous max − min (equal to the min − max) in such point $\mathbf{x}_0$: we have to prove that $\beta = 0$.
Let us assume that $\beta < 0$, i.e.

$$\max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{p}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) = \beta :$$

this implies that for every point $\mathbf{u}_2 \in U_2$ there exists a point $\mathbf{u}_1^{\mathbf{u}_2} \in U_1$ such that

$$\mathbf{p}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1^{\mathbf{u}_2}, \mathbf{u}_2) \le \beta. \tag{4.119}$$

Now, for every point $\mathbf{x} \in \mathcal{C}_{ap}$, let us denote by $\widehat{\mathbf{u}}_1^{\mathbf{x}}$ the strategy of (P) such that for every acts of (E) the trajectory can be steered from $\mathbf{x}$ into $\mathrm{int}(\mathcal{T}_0)$.



In the point $\mathbf{x}_0 \in \mathcal{B}_{ar}$. If for every point $\mathbf{u}_2 \in U_2$ there exists a point $\mathbf{u}_1^{\mathbf{u}_2} \in U_1$ such that (4.119) holds, for every choice-strategy of the second Player (E) then the first Player (P) have a control-strategy to "force" the trajectory $\widetilde{\mathbf{x}}$ from $\mathcal{B}_{ar}$ into $\mathcal{C}_{ap}$ following (locally) the direction $\dot{\widetilde{\mathbf{x}}}(0) = g(\widetilde{\mathbf{x}}_0(0), \mathbf{u}_1^{\mathbf{u}_2}, \mathbf{u}_2)$.

For every strategy $\mathbf{u}_2$ of the second Player (E), i.e. a decision rule $\boldsymbol{\nu}_2 = \boldsymbol{\nu}_2(\mathbf{x}) : \mathbb{R}^n \to U_2$, let us consider the strategy $\mathbf{u}_1^{\mathbf{u}_2}$ for the first Player (P) defined by the decision rule $\boldsymbol{\nu}_1^{\mathbf{u}_2} = \boldsymbol{\nu}_1^{\mathbf{u}_2}(\mathbf{x}) : \mathbb{R}^n \to U_1$ with $\boldsymbol{\nu}_1^{\mathbf{u}_2}(\mathbf{x}) = \mathbf{u}_1^{\boldsymbol{\nu}_2(\mathbf{x}_0)}$, for every $\mathbf{x}$. Let $\varepsilon = \varepsilon(\mathbf{u}_2) > 0$ be such that there exists the unique solution $\widetilde{\mathbf{x}}$ of the ODE

$$\begin{cases} \dot{\widetilde{\mathbf{x}}}(t) = g(\widetilde{\mathbf{x}}(t), \boldsymbol{\nu}_1^{\mathbf{u}_2}(\widetilde{\mathbf{x}}(t)), \boldsymbol{\nu}_2(\widetilde{\mathbf{x}}(t))) & \text{in } [0, \varepsilon) \\ \widetilde{\mathbf{x}}(0) = \mathbf{x}_0 \end{cases}$$

Let us notice that g in continuous with continuous derivative with respect to $\mathbf{x}$ (we are not interested to discuss the details in order to guarantee that such local solution $\widetilde{\mathbf{x}}$ there exists). Relation (4.119) implies $\widetilde{\mathbf{x}}(\epsilon) \in \mathcal{C}_{ap}$, for every strategy $\mathbf{u}_2$ of the second Player (E). Now the first Player (P) consider the strategy $\widehat{\mathbf{u}}_1^{\widetilde{\mathbf{x}}(\epsilon)}$ which transfers the point $\widetilde{\mathbf{x}}(\epsilon)$ into $\mathrm{int}(\mathcal{T}_0)$: this implies that $\mathbf{x}_0 \in \mathcal{C}_{ap}$, which contradicts the smoothness assumption on the barrier. Hence β cannot be negative. A similar proof shows that β cannot be positive. □

The fact that $\mathcal{B}_{ar}$ gives the property that without (P)'s cooperation, (E) cannot make the state cross $\mathcal{B}_{ar}$ passing from a region where $V = -1$ to a region where $V = 1$; and viceversa, without (E)'s cooperation, (P) cannot make the state cross $\mathcal{B}_{ar}$ passing from a region where $V = +1$ to a region where $V = -1$. However we have to remark that the semipermeability condition in (4.118) does not exclude a "tangential" penetration.

Let us emphasize that relation (4.118) does not imply that, for every fixed $\mathbf{x} \in \mathcal{B}_{ar}$, there exists a pair $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in U_1 \times U_2$, with $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$ that depend on $\mathbf{x}$, such that realizes the min-max in (4.118). The following notion is a further requirement on the barrier:

Definition 4.13. *Let us consider the game (4.114) and let us assume that the Isaacs' condition is satisfied. We say that a function $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*) : \mathcal{B}_{ar} \to U_1 \times U_2$ is a barrier control for $\mathcal{B}_{ar}$ if for every $\mathbf{x} \in \mathcal{B}_{ar}$ we have that*

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \mathbf{u}_2) \le 0, \qquad \forall \mathbf{u}_2 \in U_2, \tag{4.120}$$

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \boldsymbol{\nu}_2^*(\mathbf{x})) \ge 0, \qquad \forall \mathbf{u}_1 \in U_1. \tag{4.121}$$

Let us notice that the barrier control, in general, is not unique. The following remark shows the role of the barrier control[7]

Remark 4.7. *$(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*)$ is a barrier control for $\mathcal{B}_{ar}$ if and only if, for every $\mathbf{x} \in \mathcal{B}_{ar}$, the pair $(\boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}))$ realizes the* min-max *in (4.118), i.e.*

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = \min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) = 0. \tag{4.122}$$

Proof. Let $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*)$ be a barrier control for $\mathcal{B}_{ar}$ and fix $\mathbf{x} \in \mathcal{B}_{ar}$. Clearly (4.120) implies

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \le 0$$

while (4.121) implies

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \ge 0.$$

Hence $\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = 0$ and, using (4.118), we have that $(\boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}))$ realizes the min-max in (4.118).

[7] Let us notice that, due to Remark 4.7, it is possible to use (4.122) to give a different, but equivalent, definition of barrier control: in fact, in the literature appears that, a function $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*) : \mathcal{B}_{ar} \to U_1 \times U_2$ is a barrier control for $\mathcal{B}_{ar}$ if for every $\mathbf{x} \in \mathcal{B}_{ar}$ we have that (4.122) holds.

Now let us suppose that, for $\mathbf{x} \in \mathcal{B}_{ar}$ fixed, the pair $(\boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}))$ realizes the min-max in (4.118). Moreover, let us suppose that (4.120) is false, i.e. there exists $\overline{\mathbf{u}}_2 \in U_2$ such that

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \overline{\mathbf{u}}_2) > 0.$$

Since $\boldsymbol{\nu}_1^*(\mathbf{x})$ realizes the min, we obtain

$$\begin{aligned} 0 &< \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \overline{\mathbf{u}}_2) \\ &\leq \max_{\mathbf{u}_2 \in U_2} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \mathbf{u}_2) \\ &= \min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2) \end{aligned}$$

which contradicts (4.118). This gives that (4.120) is true. A similar proof gives (4.121). □

Let $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*)$ be a barrier control for $\mathcal{B}_{ar}$ (sufficiently regular), let us consider a point $\mathbf{x}_0 \in \mathcal{B}_{ar}$ and the curve $\mathbf{x} : [0, t_0] \to \mathbb{R}^n$, for some $t_0 > 0$, defined by

$$\begin{cases} \dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t))) & \text{in } [0, t_0] \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{4.123}$$

Since $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*)$ is a barrier control and using the previous ODE we have

$$0 = \mathbf{p}(\mathbf{x}(t)) \cdot g(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t))) = \mathbf{p}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t),$$

i.e. the curve $\mathbf{x}$ lies on the semipermeable surface $\mathcal{B}_{ar}$ and does not leave it. Let us show how it is possible to use this idea in order to construct the barrier: such construction is "formal", i.e. we are not interested to give the precise assumptions that the following arguments require.
Let $\mathbf{x}$ be a point in $\mathcal{B}_{ar}$. By (4.122) we have

$$\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = 0. \tag{4.124}$$

Set $\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$, $\mathbf{u}_1 = (u_{1,1}, \ldots, u_{1,k_1}) \in U_1 \subset \mathbb{R}^{k_1}$ and $\mathbf{u}_2 = (u_{2,1}, \ldots, u_{2,k_2}) \in U_2 \subset \mathbb{R}^{k_2}$. If we consider the derivative w.r.t. x_j of (4.124), we obtain

$$\begin{aligned} 0 \;=\; & \sum_{i=1}^n \frac{\partial p_i}{\partial x_j}(\mathbf{x}) g_i(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) + \sum_{i=1}^n p_i(\mathbf{x}) \frac{\partial g_i}{\partial x_j}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) + \\ & + \sum_{i=1}^n \Big[p_i(\mathbf{x}) \sum_{k=1}^{k_1} \frac{\partial g_i}{\partial u_{1,k}}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \frac{\partial \nu_{1,k}^*}{\partial x_j}(\mathbf{x}) \Big] + \\ & + \sum_{i=1}^n \Big[p_i(\mathbf{x}) \sum_{k=1}^{k_2} \frac{\partial g_i}{\partial u_{2,k}}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \frac{\partial \nu_{2,k}^*}{\partial x_j}(\mathbf{x}) \Big] \end{aligned} \tag{4.125}$$

Since $\mathcal{B}_{ar}$ is a C^2 surface, by Schwarz and (4.117) we have

$$\frac{\partial p_i}{\partial x_j}(\mathbf{x}) = \frac{\partial^2 b}{\partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial^2 b}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial p_j}{\partial x_i}(\mathbf{x}). \tag{4.126}$$

Let us notice that the third addend in (4.125) is

$$\begin{aligned} \sum_{i=1}^n \Big[p_i(\mathbf{x}) \sum_{k=1}^{k_1} \frac{\partial g_i}{\partial u_{1,k}}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \frac{\partial \nu_{1,k}^*}{\partial x_j}(\mathbf{x}) \Big] &= \sum_{k=1}^{k_1} \Big[\sum_{i=1}^n p_i(\mathbf{x}) \frac{\partial g_i}{\partial u_{1,k}}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \Big] \frac{\partial \nu_{1,k}^*}{\partial x_j}(\mathbf{x}) \\ &= \sum_{k=1}^{k_1} \frac{\partial}{\partial u_{1,k}} \Big(\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \Big) \frac{\partial \nu_{1,k}^*}{\partial x_j}(\mathbf{x}); \end{aligned}$$

since $\boldsymbol{\nu}_1^*(\mathbf{x})$ realizes the min in (4.118), i.e. the min for the function $\mathbf{u}_1 \mapsto \varphi(\mathbf{u}_1) := \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \mathbf{u}_1, \boldsymbol{\nu}_2^*(\mathbf{x}))$ (recall that $\mathbf{x}$ is fixed), then $\nabla\varphi(\boldsymbol{\nu}_1^*(\mathbf{x})) = \mathbf{0}$ and we have that

$$\frac{\partial}{\partial u_{1,k}}\Big(\mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}))\Big) = 0, \qquad \forall k = 1, \ldots, k_1.$$

Hence the third addend in (4.125) is zero; a similar argument proves that the fourth addend in (4.125) is zero. Equation (4.125) becomes, using (4.126),

$$\sum_{i=1}^{n} \frac{\partial p_j}{\partial x_i}(\mathbf{x}) g_i(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) + \sum_{i=1}^{n} p_i(\mathbf{x}) \frac{\partial g_i}{\partial x_j}(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = 0. \tag{4.127}$$

Now, let us move $\mathbf{x}$ along the barrier $\mathcal{B}_{ar}$, i.e. $\mathbf{x} = \mathbf{x}(t) \in \mathcal{B}_{ar}$ as in (4.123). Relation (4.127) becomes, using the ODE in (4.123),

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{\partial p_j}{\partial x_i}(\mathbf{x}(t)) g_i(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t))) + \sum_{i=1}^{n} p_i(\mathbf{x}(t)) \frac{\partial g_i}{\partial x_j}(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t))) \\ &= \dot{p}_j(\mathbf{x}(t)) + \frac{\partial H}{\partial x_j}(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t)), \mathbf{p}(\mathbf{x}(t))) \end{aligned}$$

since for our game the Hamiltonian is as in (4.115). Hence we have that the curve $\mathbf{x} = \mathbf{x}(t) \in \mathcal{B}_{ar}$ satisfies

$$\dot{\mathbf{p}}(\mathbf{x}(t)) = -\nabla_{\mathbf{x}} H(\mathbf{x}(t), \boldsymbol{\nu}_1^*(\mathbf{x}(t)), \boldsymbol{\nu}_2^*(\mathbf{x}(t)), \mathbf{p}(\mathbf{x}(t))). \tag{4.128}$$

Now let us consider in (4.123) $\mathbf{x}_0 \in \mathcal{BUP}$: the assumption that $\mathcal{B}_{ar}$ is natural, gives that the curve $\mathbf{x} = \mathbf{x}(t)$ "starts" in $\mathbf{x}_0$, i.e. $\mathbf{x}(0) = \mathbf{x}_0$. Recalling that $\mathcal{BUP} = \mathcal{UP} \cap \mathcal{B}_{ar}$, we have that $\mathbf{x}_0 \in \mathcal{B}_{ar}$ gives, see (4.118),

$$\min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{p}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{p}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) = 0,$$

while $\mathbf{x}_0 \in \mathcal{UP}$ gives, by definition,

$$\min_{\mathbf{u}_1 \in U_1} \max_{\mathbf{u}_2 \in U_2} \mathbf{n}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) = \max_{\mathbf{u}_2 \in U_2} \min_{\mathbf{u}_1 \in U_1} \mathbf{n}(\mathbf{x}_0) \cdot g(\mathbf{x}_0, \mathbf{u}_1, \mathbf{u}_2) \le 0.$$

If the previous inequality is strictly, $\mathbf{x}_0$ is a state of termination; this contradicts $\mathbf{x}_0 \in \mathcal{B}_{ar}$. The regularity of $\partial\mathcal{T}_0$ and $\mathcal{B}_{ar}$ gives the condition

$$\mathbf{p}(\mathbf{x}_0) = \mathbf{n}(\mathbf{x}_0). \tag{4.129}$$

Collecting the previous arguments in (4.123), (4.128) and (4.129), in order to construct the barrier we have the following;

Proposition 4.3. *Let us consider the game (4.114) and let us assume that the Isaacs' condition is satisfied. The barrier $\mathcal{B}_{ar}$ is described by the function $\mathbf{x} = \mathbf{x}(t)$ which solves the system*

$$\begin{cases} \dot{\mathbf{p}}(\mathbf{x}) = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x}), \mathbf{p}(\mathbf{x})) \\ \dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) \\ \mathbf{p}(\mathbf{x}) \cdot g(\mathbf{x}, \boldsymbol{\nu}_1^*(\mathbf{x}), \boldsymbol{\nu}_2^*(\mathbf{x})) = 0 \\ \mathbf{p}(\mathbf{x}(0)) = \mathbf{n}(\mathbf{x}(0)) \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{BUP} \end{cases} \tag{4.130}$$

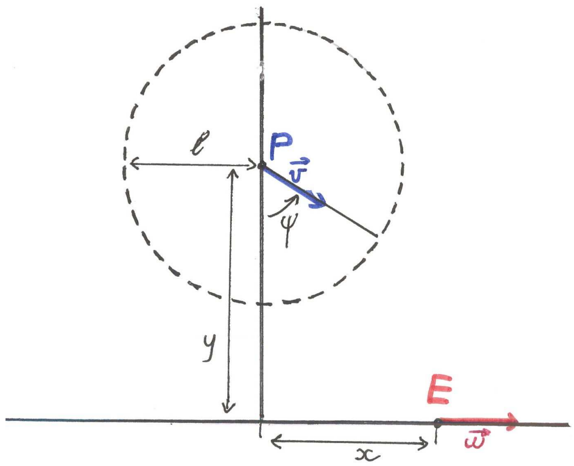
where $(\boldsymbol{\nu}_1^*, \boldsymbol{\nu}_2^*)$ is a control barrier, $\mathbf{p}(\mathbf{x}) \in \mathbb{R}^n$ is the outward normal of $\mathcal{C}_{ap}$ in the point $\mathbf{x} \in \mathcal{B}_{ar}$ as in (4.117).

4.5.1 Interception of a straight flying evader

When can an interceptor be successful against a faster attacking craft which travels a fixed straight course? This model is in the book of Isaacs (see section 8.6 in [10]).

Here P moves in the plane with a motion and unit speed, while E is bound to a line moves with speed w ($w > 0$ fixed), and merely can select for his strategy one of the two possible directions of travel. Capture occurs when $|PE| < l$, where $l > 0$ fixed. In view of P's unquestioned ability to capture when $w < 1$, this case is trivial (please, solve it !). Our interest is in the conditions which make possible the success of a slower pursuer with $w > 1$.

Let us pass to the details. The Pursuer P is free to move itself in the half-plane $y \geq 0$: the velocity is $\overrightarrow{v}(t)$ with modulo $v = 1$, the angle of the velocity w.r.t. the y-axis is ψ. The Evader E, constrained on the x-axis, can only controls the direction $\varphi \in \{\pm 1\}$ of the velocity is $\overrightarrow{w}(t)$, with modulo $w > 1$ fixed.



The dynamics is

$$\begin{cases} \dot{x} = \omega\varphi - \sin\psi \\ \dot{y} = -\cos\psi \end{cases}$$

and the target set $\mathcal{T} = [0, \infty) \times \mathcal{T}_0$ is such that

$$\mathcal{T}_0 = \{(x, y) \in \mathcal{G}_0 : y \geq 0, \ \sqrt{x^2 + y^2} \leq l\}.$$

The boundary of $\mathcal{T}_0$ is smooth and

$$\partial\mathcal{T}_0 = \left\{ \mathbf{x} = l(\sin\alpha, \cos\alpha) : \ \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\};$$

the outward normal of $\mathcal{T}_0$ in $\mathbf{x} \in \partial\mathcal{T}_0$ is $\mathbf{n}(\mathbf{x}) = (\sin\alpha, \cos\alpha)$. The game hence is

$$\begin{cases} Pursuer: \ \min_{\psi} J(\psi, \varphi), \qquad Evader: \ \max_{\varphi \in \{-1,+1\}} J(\psi, \varphi) \\ J(\psi, \varphi) = \begin{cases} -1 & \text{if } \exists\, t \geq 0 \ \text{ s.t. } \ \|(x(t), y(t))\|_2 < l \\ +1 & \text{otherwise} \end{cases} \\ \dot{x} = \omega\varphi - \sin\psi \\ \dot{y} = -\cos\psi \\ (x(0), y(0)) = (x_0, y_0), \qquad y_0 \geq 0 \end{cases}$$

We are in a pursuit-evasion game as in (4.114).

The game set is $\mathcal{G} = [0, \infty) \times \mathcal{G}_0$. In order to describe $\mathcal{G}_0$, let us fix $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^+$: it is easy to see that, first, choosing in the dynamics $\psi = 0$, for every φ, we move (x_0, y_0) to the point $(x_1, 0)$, for some x_1; secondly, choosing $\psi = \pm\pi/2$ and $\varphi = \pm 1$ depending on $\mathrm{sgn}(x_1)$, the dynamics moves the point $(x_1, 0)$ to the origin $(0, 0) \in \mathrm{int}(\mathcal{T}_0)$. Hence

$$\mathcal{G}_0 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}.$$

The Hamiltonian is $H(x,y,\psi,\varphi,\lambda_1,\lambda_2)=\lambda_1(\omega\varphi-\sin\psi)-\lambda_2\cos\psi$ and it is easy to verify that the Isaacs' condition is satisfied. Let us looking for the usable part $\mathcal{UP}$, i.e.

$$\mathcal{UP}=\{\mathbf{x}=l(\sin\alpha,\cos\alpha)\in\partial\mathcal{T}_0:\ \min_{\psi}\ \max_{\varphi\in\{-1,+1\}}\mathbf{n}(\mathbf{x})\cdot g(\mathbf{x},\psi,\varphi)\le 0\};$$

hence we have to find $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$ such that, since w is positive,

$$\min_{\psi}(-\sin\alpha\sin\psi-\cos\alpha\cos\psi)+w\max_{\varphi\in\{-1,+1\}}\varphi\sin\alpha\le 0.$$

This is equivalent to $-1+w|\sin\alpha|\le 0$. Hence, recalling that $w>1$,

$$\mathcal{UP}=\left\{\mathbf{x}=l(\sin\alpha,\cos\alpha):\ |\sin\alpha|\le\frac{1}{w}\right\}.$$

The boundary of the usable part is

$$\mathcal{BUP}=\left\{\mathbf{x}=l(\sin\alpha,\cos\alpha):\ |\sin\alpha|=\frac{1}{w}\right\}=\left\{\mathbf{x}_0^+:=l(\sin\alpha^+,\cos\alpha^+),\ \mathbf{x}_0^-:=l(\sin\alpha^-,\cos\alpha^-)\right\},$$

where $-\pi/2<\alpha^-<0<\alpha^+<\pi/2$ such that $|\sin\alpha^{\pm}|=\frac{1}{w}$.

Let us construct the barrier $\mathcal{B}_{ar}$, i.e. the curve $\mathbf{x}(t)=(x(t),y(t))$ which is solution of the system (4.130), i.e.

$$\begin{cases}\dot p_1(x,y)=-\dfrac{\partial H}{\partial x}(x,y,\psi^*(x,y),\varphi^*(x,y),p_1(x,y),p_2(x,y))=0\\ \dot p_2(x,y)=-\dfrac{\partial H}{\partial y}(x,y,\psi^*(x,y),\varphi^*(x,y),p_1(x,y),p_2(x,y))=0\\ \dot x=\omega\varphi^*(x,y)-\sin\psi^*(x,y)\\ \dot y=-\cos\psi^*(x,y)\\ p_1(x,y)(\omega\varphi^*(x,y)-\sin\psi^*(x,y))-p_2(x,y)\cos\psi^*(x,y)=0\\ \mathbf{p}(x(0),y(0))=\mathbf{n}(x(0),y(0))\\ (x(0),y(0))\in\mathcal{BUP}\end{cases}\tag{4.131}$$

recalling that $\mathbf{p}(\mathbf{x})=(p_1(x,y),p_2(x,y))$ is the outward normal of $\mathcal{C}_{ap}$ in $\mathbf{x}=(x,y)\in\mathcal{B}_{ar}$ and denoting by $(\boldsymbol{\nu}_1^*(\mathbf{x}),\boldsymbol{\nu}_2^*(\mathbf{x}))=(\psi^*(x,y),\ \varphi^*(x,y))$ the control barrier who realizes the following $\min-\max$

$$\min_{\psi}\ \max_{\varphi\in\{-1,+1\}}\mathbf{p}(x,y)\cdot g(x,y,\psi,\varphi)=\min_{\psi}(-p_1(x,y)\sin\psi-p_2(x,y)\cos\psi)+\omega\max_{\varphi\in\{-1,+1\}}p_1(x,y)\varphi,$$

i.e.

$$(\cos\psi^*(x,y),\sin\psi^*(x,y))=(p_2(x,y),p_1(x,y)),\qquad \varphi^*(x,y)=\mathrm{sgn}(p_1(x,y)).\tag{4.132}$$

Let us consider $\mathbf{x}_0^+\in\mathcal{BUP}$. The first two equations of the system (4.131) give that $\mathbf{p}(x,y)$ is a constant; hence the barrier is part of lines. Since $\mathcal{B}_{ar}$ starts from the point $\mathbf{x}_0^+$ and it is tangent to $\partial\mathcal{T}_0$ in such point (see the sixth condition in the (4.131)), we have that the equation of the barrier is

$$\mathbf{x}(t)=(x(t),y(t))=tk(-\cos\alpha^+,\sin\alpha^+)+l(\sin\alpha^+,\cos\alpha^+);\tag{4.133}$$

for some non zero constant k. Now, by noticing that $\mathcal{B}_{ar}$ divides the escape states $\mathcal{E}_{sc}$ to the capture states $\mathcal{C}_{ap}$ and that $\mathcal{UP}\setminus\mathcal{BUP}$ is inside $\mathcal{C}_{ap}$, in (4.133) we have only to consider $t\ge 0$ and $k>0$.

The outward normal of $\mathcal{C}_{ap}$ along $\mathcal{B}_{ar}$ in the point $\mathbf{x}(t)=(x(t),y(t))$ in (4.133) is[8]

$$\mathbf{p}(\mathbf{x}(t))=(p_1(x(t),y(t)),\ p_2(x(t),y(t)))=(\sin\alpha^+,\cos\alpha^+):\tag{4.134}$$

hence in (4.132) we obtain

$$\psi^*(x,y)=\alpha^+,\qquad \varphi^*(x,y)=1.\tag{4.135}$$

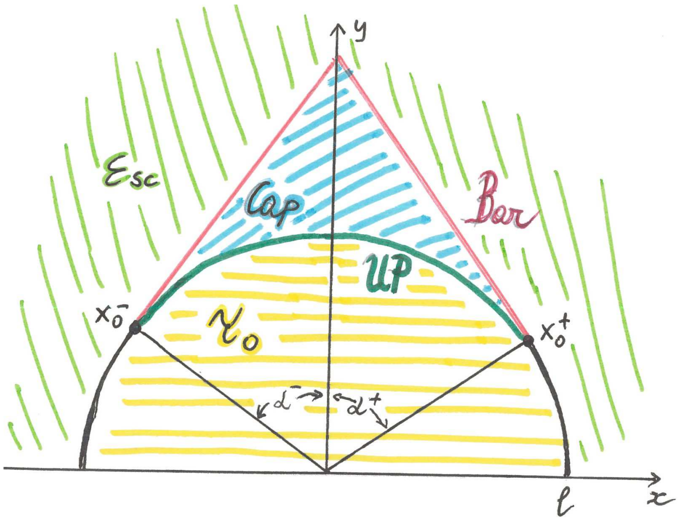
[8]Note that, by (4.133), $\dot{\mathbf{x}}(t)=k(-\cos\alpha^+,\sin\alpha^+)$ and by (4.134) we have $\dot{\mathbf{x}}(t)\perp\mathbf{p}(\mathbf{x}(t))$.

Let us verify that the third, the fourth and the fifth equations in (4.131) are satisfied: using (4.133)–(4.134) we obtain that the mentioned relations in (4.131)

$$
\begin{aligned}
&-k\cos\alpha^{+} = \omega 1 - \sin\alpha^{+} \\
&k\sin\alpha^{+} = -\cos\alpha^{+} \\
&\sin\alpha^{+}(\omega 1 - \sin\alpha^{+}) - \cos\alpha^{+}\cos\alpha^{+} = 0
\end{aligned}
$$

are true, choosing $k = -\mathrm{cotan}\,\alpha^{+}$.

A similar argument holds for $\mathbf{x}_0^-$. Hence we obtain:



Appendix A

Optimal control tools

Let us consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(\mathbf{x}(t_1)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}} \; J(\mathbf{u}), \\ \mathcal{C} = \{\mathbf{u} : [t_0, t_1] \to U \subseteq \mathbb{R}^k, \;\; \mathbf{u} \text{ admissible}\} \end{cases} \tag{A.1}$$

where t_0 and t_1 are fixed.

A.1 Variational approach

We define the *Hamiltonian function* $H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ for the problem (A.1) by

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}).$$

The following result is fundamental:

Theorem A.1 (Pontryagin). *Let us consider the problem (A.1) with $f \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$ and $g \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$.*
Let $\mathbf{u}^*$ be an optimal control and $\mathbf{x}^*$ be the associated trajectory.
Then there exists a continuous multiplier $\boldsymbol{\lambda}^* : [t_0, t_1] \to \mathbb{R}^n$ such that

i) (Pontryagin Maximum Principle) for all $\tau \in [t_0, t_1]$ we have

$$\mathbf{u}^*(\tau) \in \arg\max_{\mathbf{v} \in U} H(\tau, \mathbf{x}^*(\tau), \mathbf{v}, \lambda_0^*, \boldsymbol{\lambda}^*(\tau));$$

ii) (adjoint equation) in $[t_0, t_1]$ we have $\dot{\boldsymbol{\lambda}}^* = -\nabla_{\mathbf{x}} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)$;

iii) (transversality condition) $\boldsymbol{\lambda}^*(t_1) = \nabla_{\mathbf{x}} \psi(\mathbf{x}(t_1))$;

A first sufficient condition is the following

Theorem A.2 (Mangasarian). *Let us consider the maximum problem (A.1) with $f \in C^1$ and $g \in C^1$. Let the control set U be convex. Let $\mathbf{u}^*$ be a normal extremal control, $\mathbf{x}^*$ the associated trajectory and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ the associated multiplier (as in theorem A.1).*
Consider the Hamiltonian function H and let us suppose that

v) the function $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*(t))$ is, for every $t \in [t_0, t_1]$, concave.

Then $\mathbf{u}^*$ *is optimal.*

Another and useful sufficient condition is due to Arrow.

Theorem A.3 (Arrow). *Let us consider the maximum problem (A.1) with* $f \in C^1$ *and* $g \in C^1$. *Let* $\mathbf{u}^*$ *be a normal extremal control,* $\mathbf{x}^*$ *be the associated trajectory and* $\boldsymbol{\lambda}^*$ *be the associated multiplier.*
Let us suppose that exists the maximized Hamiltonian function $H^0 : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by*

$$H^0(t, \mathbf{x}, \boldsymbol{\lambda}) = \max_{\mathbf{u} \in U} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}), \tag{A.2}$$

where $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u})$ *is the Hamiltonian. Let us suppose that, for every* $t \in [t_0, t_1] \times \mathbb{R}^n$, *the function*

$$\mathbf{x} \mapsto H^0(t, \mathbf{x}, \boldsymbol{\lambda}^*(t))$$

is concave. Then $\mathbf{u}^*$ *is optimal.*

A.1.1 Infinite horizon problems

Let us consider the problem:

$$\begin{cases} \max_{\mathbf{u} \in \mathcal{C}} \int_{t_0}^{\infty} f(t, \mathbf{x}, \mathbf{u}) \, \mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \lim_{t \to \infty} x_i(t) = \beta_i, & \text{for } 1 \le i \le n' \\ \lim_{t \to \infty} x_i(t) \text{ free} & \text{for } n' < i \le n \\ \mathcal{C} = \{\mathbf{u} : [t_0, \infty) \to U \subseteq \mathbb{R}^k, \ \mathbf{u} \text{ admissible}\} \end{cases} \tag{A.3}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are fixed in $\mathbb{R}^n$. We give a sufficient condition in the spirit of the theorem of Mangasarian:

Theorem A.4. *Let us consider the infinite horizon maximum problem (A.3) with* $f \in C^1$ *and* $g \in C^1$. *Let the control set* U *be convex. Let* $\mathbf{u}^*$ *be a normal extremal control,* $\mathbf{x}^*$ *the associated trajectory and* $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ *the associated multiplier, i.e. the tern* $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)$ *satisfies the PMP and the adjoint equation.*
Suppose that

v) the function $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*)$ *is, for every* $t \in [t_0, \infty)$, *concave,*

vi) for all admissible trajectory $\mathbf{x}$,

$$\lim_{t \to \infty} \boldsymbol{\lambda}^*(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \ge 0. \tag{A.4}$$

Then $\mathbf{u}^*$ *is optimal.*

Remark A.1. *Suppose that in the problem (A.3) we have only a condition of the type* $\lim_{t \to \infty} x_i(t) = \beta_i$. *Suppose that there exists a constant* c *such that*

$$|\boldsymbol{\lambda}^*(t)| \le c, \qquad \forall t \ge \tau \tag{A.5}$$

for some τ, *then the transversality condition in (A.4) holds.*

In many problems of economic interest, future values of income and of expenses are discounted: if $r > 0$ is the discount rate, we have the problem

$$\begin{cases} J(\mathbf{u}) = \int_{t_0}^{\infty} e^{-rt} f(t, \mathbf{x}, \mathbf{u}) \, \mathrm{d}t \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \max_{\mathbf{u} \in \mathcal{C}} J(\mathbf{u}) \end{cases} \tag{A.6}$$

Let us define the *current Hamiltonian* H^c as

$$H^c(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}_c) = f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}_c \cdot g(t, \mathbf{x}, \mathbf{u}),$$

where $\boldsymbol{\lambda}_c$ is the *current multiplier*. Clearly we obtain

$$\begin{aligned} H^c &= e^{rt} H \\ \boldsymbol{\lambda}_c^* &= e^{rt} \boldsymbol{\lambda}^*. \end{aligned} \tag{A.7}$$

A necessary condition for the problem (A.6) is

Remark A.2.

$$\begin{aligned} \mathbf{u}^* &\in \arg\max_{\mathbf{v} \in U} H^c(t, \mathbf{x}^*, \mathbf{v}, \boldsymbol{\lambda}_c^*) \\ \dot{\boldsymbol{\lambda}}_c^* &= r\boldsymbol{\lambda}_c^* - \nabla_{\mathbf{x}} H^c(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}_c^*) \end{aligned}$$

In order to use a necessary condition of optimality as in Theorem A.4, we note that (A.7) implies that the concavity of

$$(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^*(t)), \qquad \forall t$$

is equivalent to the concavity

$$(\mathbf{x}, \mathbf{u}) \mapsto H^c(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}_c^*(t)), \qquad \forall t$$

A.2 Dynamic Programming

Let us consider the problem for the problem (A.1); we define the *Hamiltonian of Dynamic Programming* $H_{DP} : [t_0, t_1] \times \mathbb{R}^{2n} \to (-\infty, +\infty]$ defined by

$$H_{DP}(t, \mathbf{x}, \boldsymbol{\lambda}) = \max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) \tag{A.8}$$

We have the following necessary condition

Theorem A.5. *Let us consider the problem (A.1) and let us suppose that for every $(\tau, \boldsymbol{\xi}) \in [t_0, t_1] \times \mathbb{R}^n$ there exists the optimal control ${\mathbf{u}^*}_{\tau,\boldsymbol{\xi}}$ for the problem with initial data $\mathbf{x}(\tau) = \boldsymbol{\xi}$. Let V be the value function for the problem (A.1) and let V be differentiable. Then, for every $(t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n$, we have*

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, \mathbf{x}) + H_{DP}(t, \mathbf{x}, \nabla_{\mathbf{x}} V(t, \mathbf{x})) = 0 & \text{for } (t, \mathbf{x}) \in [t_0, t_1] \times \mathbb{R}^n \\ V(t_1, \mathbf{x}) = \psi(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^n \end{cases} \tag{A.9}$$

We give a sufficient condition for a more general problem: let us consider the problem

$$\begin{cases} J(\mathbf{u}) = \displaystyle\int_{t_0}^{T} f(t, \mathbf{x}, \mathbf{u})\, \mathrm{d}t + \psi(T, \mathbf{x}(T)) \\ \dot{\mathbf{x}} = g(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ (T, \mathbf{x}(T)) \in S \\ \displaystyle\max_{\mathbf{u} \in \mathcal{C}_{t_0, \boldsymbol{\alpha}}} J(\mathbf{u}), \end{cases} \tag{A.10}$$

with a control set $U \subset \mathbb{R}^k$, with the target set $S \subset (t_0, \infty) \times \mathbb{R}^n$. Let us consider the *reachable set* for the target set S defined by

$$R(S) = \{(\tau, \boldsymbol{\xi}) : \ \mathcal{C}_{\tau, \boldsymbol{\xi}} \neq \emptyset\},$$

i.e. as the set of the points $(\tau, \boldsymbol{\xi})$ from which it is possible to reach the terminal target set S with some trajectory.

Theorem A.6. *Let us consider the problem (A.10) with S closed. Let $W : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 solution of the BHJ equation*

$$\frac{\partial W}{\partial t}(t, \mathbf{x}) + \max_{\mathbf{v} \in U} \Big(f(t, \mathbf{x}, \mathbf{v}) + \nabla_{\mathbf{x}} W(t, \mathbf{x}) \cdot g(t, \mathbf{x}, \mathbf{v}) \Big) = 0,$$

for every $(t, \mathbf{x})$ in the interior of the reachable set $R(S)$. Suppose that the final condition

$$W(t, \mathbf{x}) = \psi(t, \mathbf{x}), \qquad \forall (t, \mathbf{x}) \in S \tag{A.11}$$

holds. Let $(t_0, \boldsymbol{\alpha})$ be in the interior of $R(S)$ and let $\mathbf{u}^* : [t_0, T^*] \to U$ be a control in $\mathcal{C}_{t_0, \boldsymbol{\alpha}}$ with corresponding trajectory $\mathbf{x}^*$ such that

$$\frac{\partial W}{\partial t}(t, \mathbf{x}^*(t)) + f(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) + \nabla_{\mathbf{x}} W(t, \mathbf{x}^*(t)) \cdot g(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) = 0,$$

for every $\in [t_0, T^*]$. Then $\mathbf{u}^*$ is the optimal control with exit time T^*.

- **Multiplier as shadow price**

Theorem A.7. *Let $\mathbf{x}^*_{t_0, \boldsymbol{\alpha}}$ be the optimal trajectory, $\boldsymbol{\lambda}^*_{t_0, \boldsymbol{\alpha}}$ be the optimal multiplier and let V be the value function for the problem A.1 with initial data $\mathbf{x}(t_0) = \boldsymbol{\alpha}$. If V is differentiable, then*

$$\nabla_{\mathbf{x}} V(t, \mathbf{x}^*_{t_0, \boldsymbol{\alpha}}(t)) = \boldsymbol{\lambda}^*_{t_0, \boldsymbol{\alpha}}(t), \tag{A.12}$$

for every $t \in [t_0, t_1]$.

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