Lagrangian formalism

$\mathcal{L}(q, \dot{q}, t)$ q: generalized coordinate

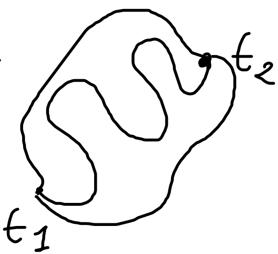
$\dot{q} = \frac{dq}{dt}$

Cartesian system q: x, y, z

Cylindrical system q: r, θ, z

Spherical system: q: r, θ, φ

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$$



$\frac{dS}{dt} = 0$ d'Alembert principle

Lagrange equations

$$\mathcal{L}(q_1 \dots q_N, \dot{q}_1 \dots \dot{q}_N, t)$$

q_j

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \qquad j = 1 \dots N$$

N: # degrees of freedom

<u>Exact constants of motion</u>

If $\underbrace{\frac{\partial \mathcal{L}}{\partial q_j} = 0}_{\text{eg. if } q_j \text{ does } \underline{\text{not}} \text{ appear in } \mathcal{L}}$ then $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0$

$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$ = const of motion

$p_j = \partial \mathcal{L} / \partial \dot{q}_j$ <u>canonical momentum</u>

Hamiltonian formalism

$j = 1 \dots N$ degrees of freedom

$$H = \sum_{j=1}^{N} P_j \dot{Q}_j - \mathcal{L}(Q, \dot{Q}, t)$$

$$\dot{Q}_j = \frac{\partial H}{\partial P_j} \qquad \dot{P}_j = -\frac{\partial H}{\partial Q_j}$$

$$\frac{dH}{dt} = \sum_{j=1}^{N} \cancel{\overset{\frac{\partial \mathcal{L}}{\partial Q_j}}{\dot{P}_j} \dot{Q}_j} + \cancel{\underset{\frac{\partial \mathcal{L}}{\partial \dot{Q}_j}}{P_j} \ddot{Q}_j} - \left[\sum_{j=1}^{N} \left(\cancel{\frac{\partial \mathcal{L}}{\partial Q_j} \dot{Q}_j} + \cancel{\frac{\partial \mathcal{L}}{\partial \dot{Q}_j} \ddot{Q}_j} \right) + \frac{\partial \mathcal{L}}{\partial t} \right] = -\frac{\partial \mathcal{L}}{\partial t}$$

Lagrange equations:

$$\frac{d}{dt}\Big(\overbrace{\frac{\partial \mathcal{L}}{\partial \dot{Q}_j}}^{P_j} \Big) = \frac{\partial \mathcal{L}}{\partial Q_j} \Rightarrow \dot{P}_j = \frac{\partial \mathcal{L}}{\partial Q_j}$$

If $\frac{\partial \mathcal{L}}{\partial t} = 0$ (if $\mathcal{L}$ does not have an explicit dependence on t)

then

$$\frac{dH}{dt} = 0 \Rightarrow H \text{ is a constant of motion}$$

Lagrangian and Hamiltonian of a q in $\underline{E}(\underline{x}, t)$ and $\underline{B}(\underline{x}, t)$?

$$\mathcal{L} = \frac{mv^2}{2} + q\,\underline{v} \cdot \underline{A}(\underline{x}, t) - q\,\phi(\underline{x}, t)$$

$\underline{A}(\underline{x}, t)$: vector potential

$\phi(\underline{x}, t)$: scalar potential

$$\underline{B}(\underline{x}, t) = \underline{\nabla} \times \underline{A}(\underline{x}, t)$$

$$\underline{E}(\underline{x}, t) = -\underline{\nabla}\phi(\underline{x}, t) - \frac{\partial \underline{A}}{\partial t}(\underline{x}, t)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}_j} - \frac{\partial \mathcal{L}}{\partial Q_j} = 0 \Rightarrow \underline{F} = q\,(\underline{v}\times\underline{B} + \underline{E})$$

(Lorentz Force)

$Q_j = v_x, v_y, v_z \quad (\dot{x}_j)$

$Q_j = x, y, z \quad (x_j)$

$$\mathcal{L} = \frac{mv^2}{2} + q\,\underline{A}\cdot(\underline{x},t)\cdot\underline{v} - q\,\phi(\underline{x},t)$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = 0 + q\frac{\partial}{\partial x_j}\sum_{i=1}^{3} A_i(\underline{x},t)\,v_i - q\frac{\partial\phi}{\partial x_j}(\underline{x},t) =$$

$$= q\sum_{i=1}^{3}\frac{\partial A_i}{\partial x_j}(\underline{x},t)\,v_i - q\frac{\partial\phi}{\partial x_j}(\underline{x},t)$$

canonical momentum for a q in $\begin{cases}\underline{E}(x,t)\\ \underline{B}(\underline{x},t)\end{cases}$

$\underline{P} = m\underline{v} + q\,\underline{A}$

$$P_j = \frac{\partial \mathcal{L}}{\partial \dot{x}_j} = \frac{m}{2}\frac{\partial}{\partial \dot{x}_j}\sum_{i=1}^{3}\dot{x}_i^2 + q\frac{\partial}{\partial \dot{x}_j}\sum_{i=1}^{3}A_i(\underline{x},t)\cdot\dot{x}_i + 0 = \frac{m}{\cancel{2}}\,\cancel{2}\dot{x}_j + q A_j\cdot(\underline{x},t)$$

$$= m\dot{x}_j + q A_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt}\left(m\dot{x}_j + q A_j(\underline{x},t) \right)$$

$$= m\ddot{x}_j + q\sum_{i=1}^{3}\frac{\partial A_j}{\partial x_i}\dot{x}_i + q\frac{\partial A_j}{\partial t}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

$$m\ddot{x}_j + q\sum_{i=1}^{3}\frac{\partial A_j}{\partial x_i}\dot{x}_i + q\frac{\partial A_j}{\partial t} - q\sum_{i=1}^{3}\frac{\partial A_i}{\partial x_j}v_i + q\frac{\partial \phi}{\partial x_j} = 0$$

$$m\ddot{x}_j = q\left(-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t}\right) + q\left(\sum_{i=1}^{3}\frac{\partial A_i}{\partial x_j}v_i - \frac{\partial A_j}{\partial x_i}v_i\right)$$

$$m\ddot{x}_j = q\left(-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t}\right) + \underbrace{q\sum_{i=1}^{3} v_i\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)}$$

$$m\frac{d\underline{v}}{dt} = q\left(\underline{E} + \underline{v}\times\underline{B}\right) \qquad E_j \qquad q\left(\underline{v}\times\underline{B}\right)$$

$$m\frac{dv_j}{dt} = q\left(E_j + (\underline{v}\times\underline{B})_j\right)$$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A} \Rightarrow E_j = -\frac{\partial\phi}{\partial x_j} - \frac{\partial A_j}{\partial t}$$

$$\underline{v}\times\underline{B} = \underline{v}\times\left(\underline{\nabla}\times\underline{A}\right)$$

$\underline{v} \times (\underline{\nabla} \times \underline{A})$ $\qquad$ x comp.

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \left[\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \right] =$$

$$= \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} v_y (\partial_x A_y - \partial_y A_x) - v_z (\partial_z A_x - \partial_x A_z) \\ \text{- - - -} \\ \text{- - - -} \end{pmatrix}$$

$$\sum_{i=1}^{3} v_i \left(\frac{\partial A_i}{\partial x} - \frac{\partial A_x}{\partial x_i} \right) = \cancel{v_x \frac{\partial A_x}{\partial x}} - \cancel{v_x \frac{\partial A_x}{\partial x}} + v_y \frac{\partial A_y}{\partial x} - v_y \frac{\partial A_x}{\partial y} =$$

$$+ v_z \frac{\partial A_z}{\partial x} - v_z \frac{\partial A_x}{\partial z}$$

Hamiltonian

$$H = \underline{P}\cdot\underline{v} - \mathcal{L} = mv^2 + q\,\underline{A}\cdot\underline{v} - \left(\frac{mv^2}{2} + q\,\underline{A}\cdot\underline{v} + q\phi\right) =$$

$$\underline{P} = m\underline{v} + q\underline{A}$$

$$H = \frac{mv^2}{2} + q\phi =$$

$$= \frac{m}{2}\left(\underline{P} - q\underline{A}\right)^2 + q\phi = H(\underline{x}, \underline{P}, t)$$

$(\underline{x}, \underline{P})$