Linear mixed models
Part I

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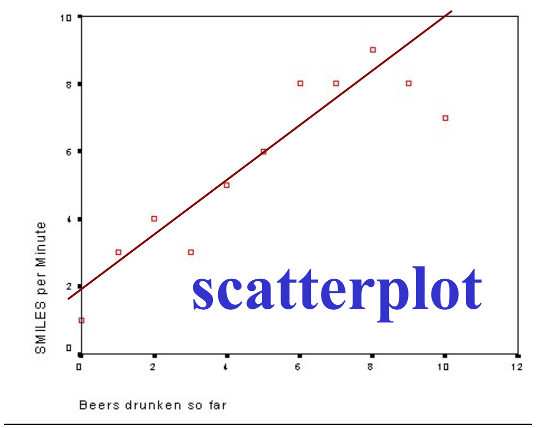
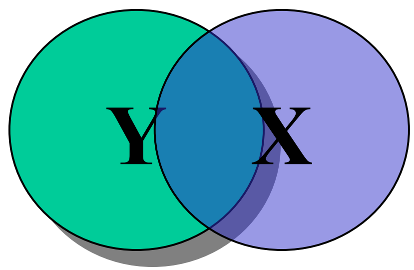
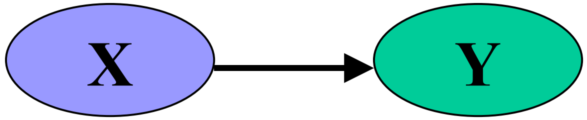
Introduction

A simple **statistical model** is an efficient and concise representation of the data describing an empirical phenomenon

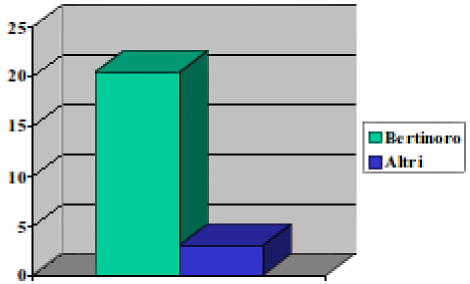




Path Diagram



Difference in mean

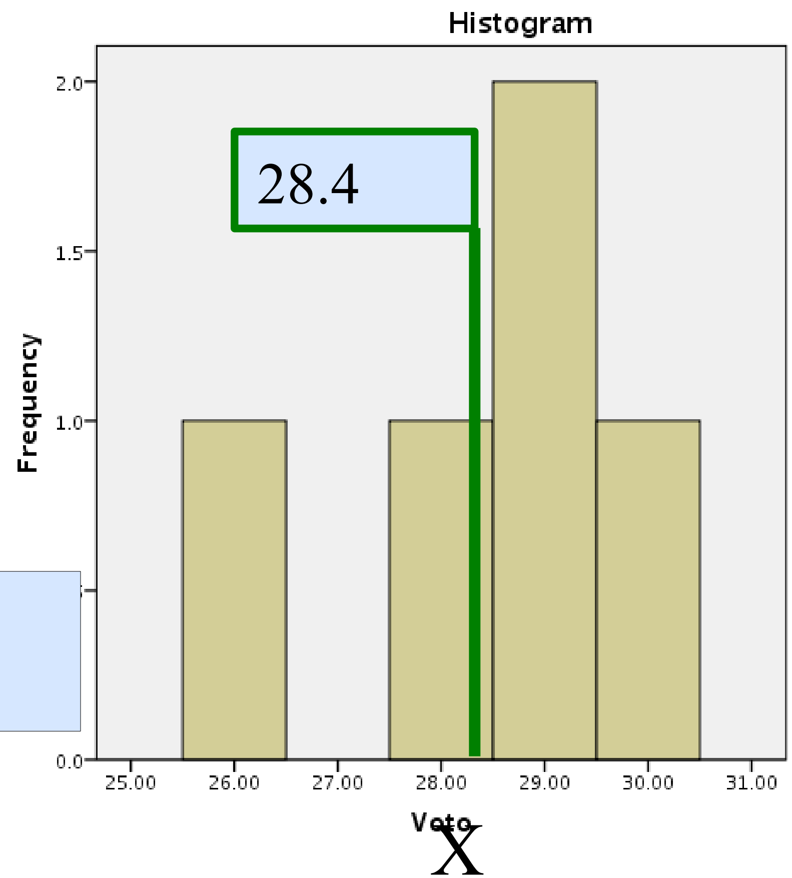


Introduction

The statistical model and its representation is aimed at:

- Efficient and Concise Description
- Prediction of future
- Generalization to a population

That is: understanding of main trends in data

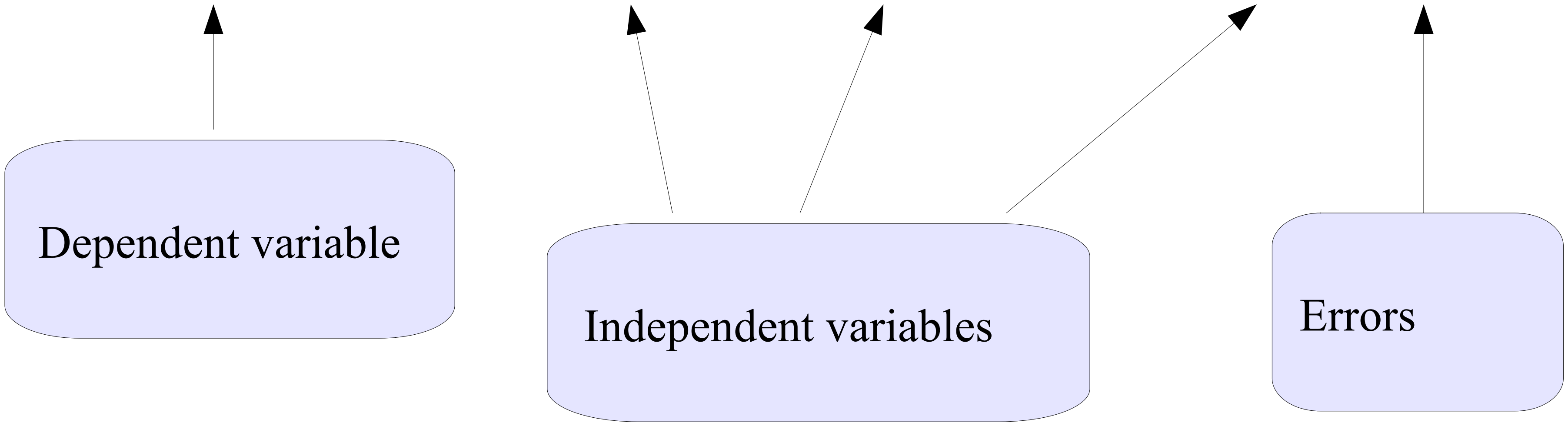


GLM

- The majority of the statistical technique that we use belong to one single **general model**

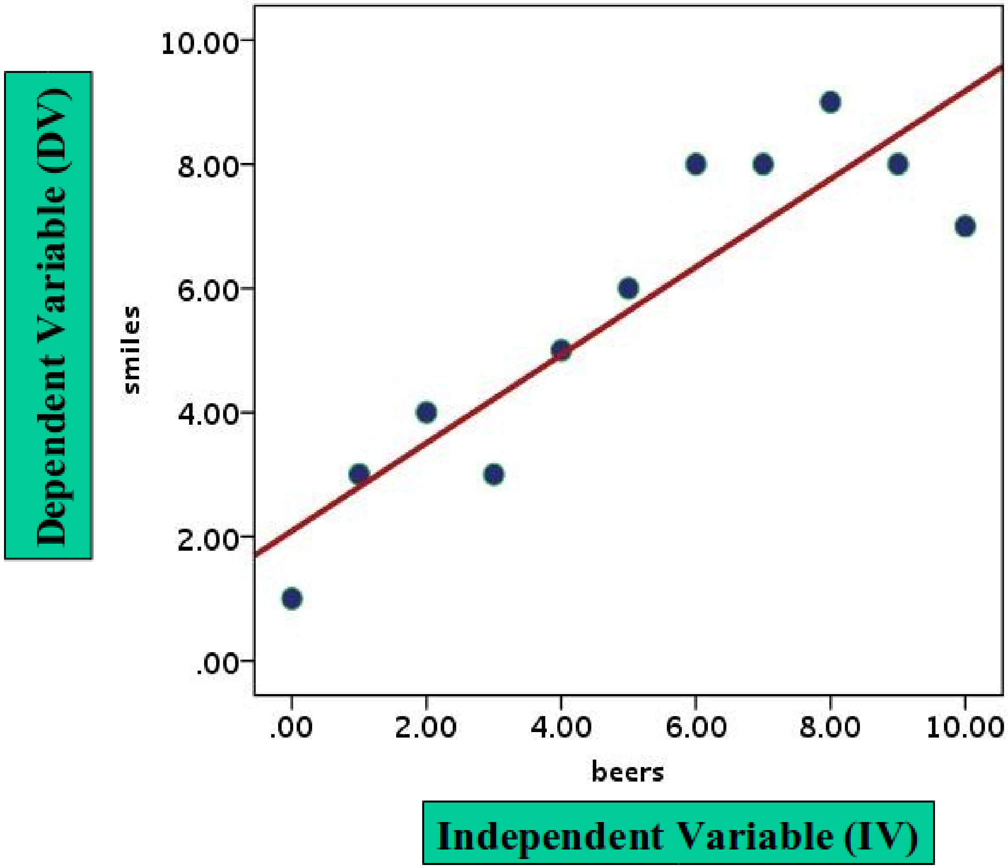
General Linear Model

$$y_i = a + b_1 \cdot x_{1i} + b_2 \cdot x_{2i} + \, .. b_k \cdot x_{ki} + e_i$$



GLM Example

The aim of regression analysis is to fit the data using a function



For most applications, we just need a linear function: straight line

$$y_i = a + b \cdot x_i + e_i$$

$$\hat{y}_i = a + b \cdot x_i$$

GLM

pros

- It allows to estimate the effects of one or several IVs on a DV
- It can be applied in many different research problems
- It allows to estimate many different types of effects

cons

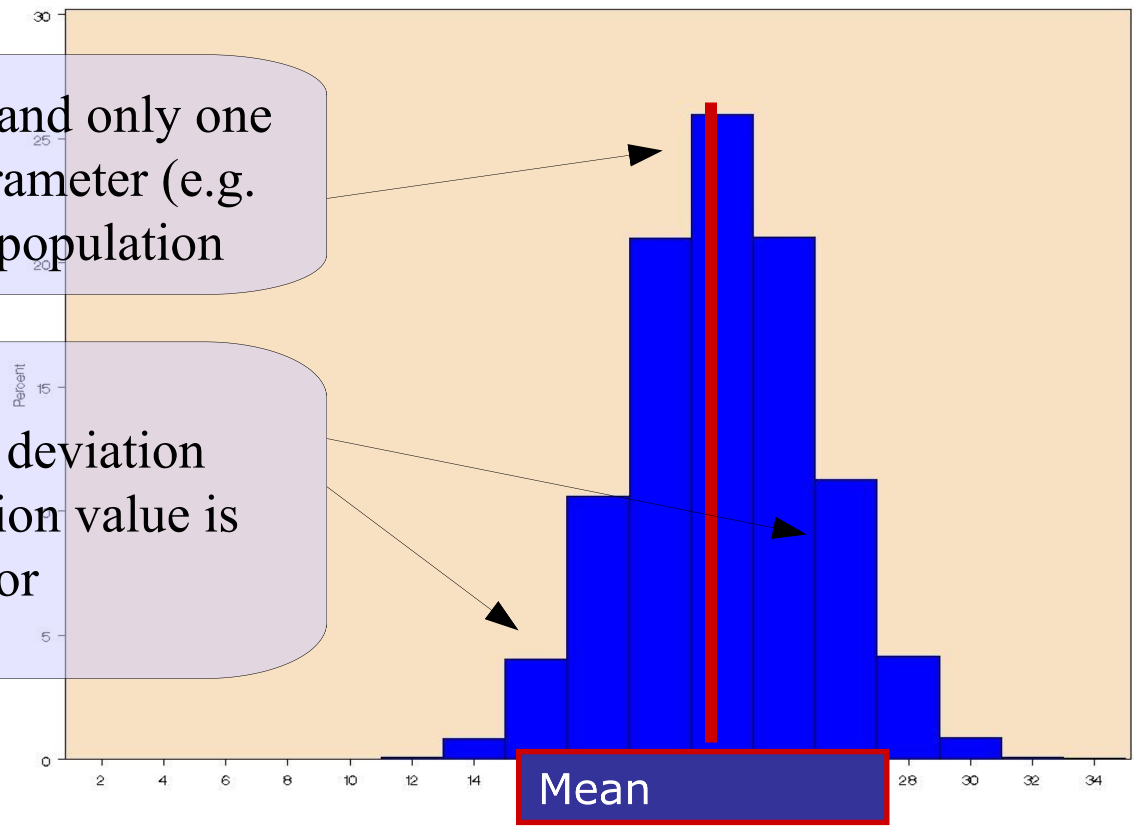
- It assumes data have a very simple structure
- It applies on a limited amount of dependent variables (continuous or semi-continuous DV)

GLM Assumptions

$$y_i = a + e_i$$

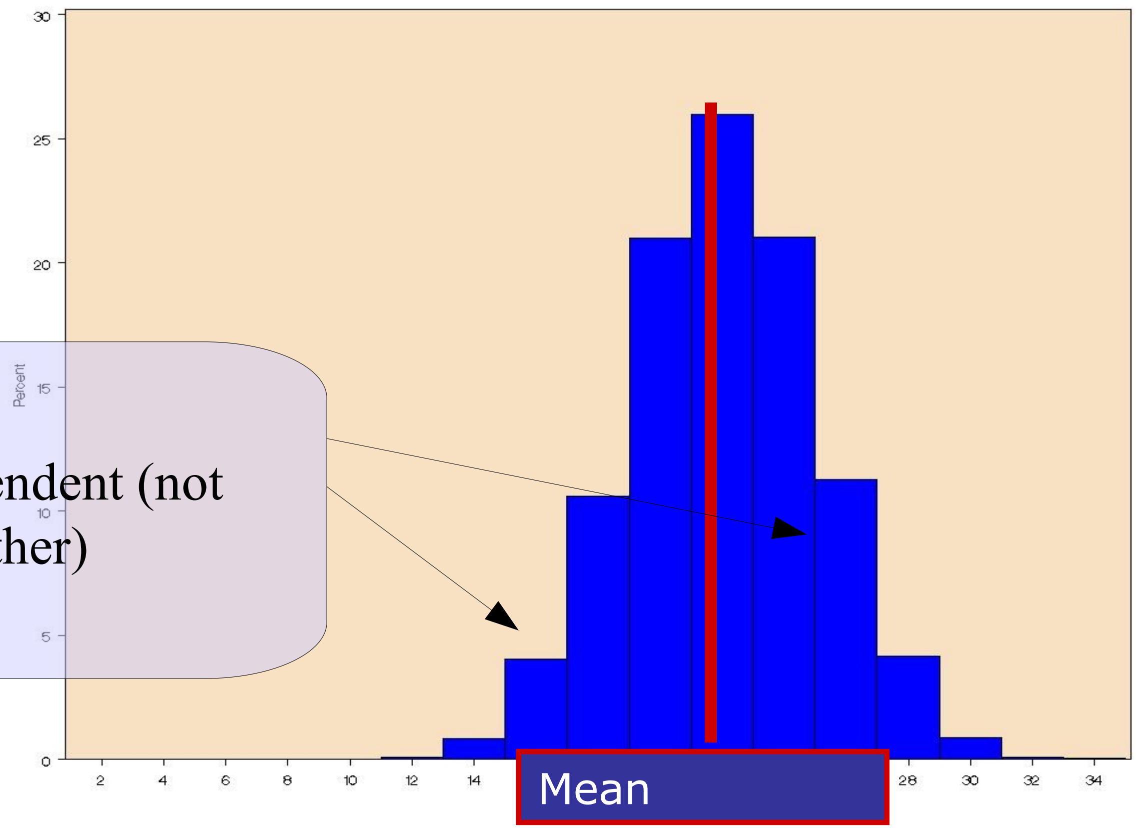
1) The exist one and only one value of each parameter (e.g. the mean) in the population

2) Any observed deviation from the population value is deemed to be error



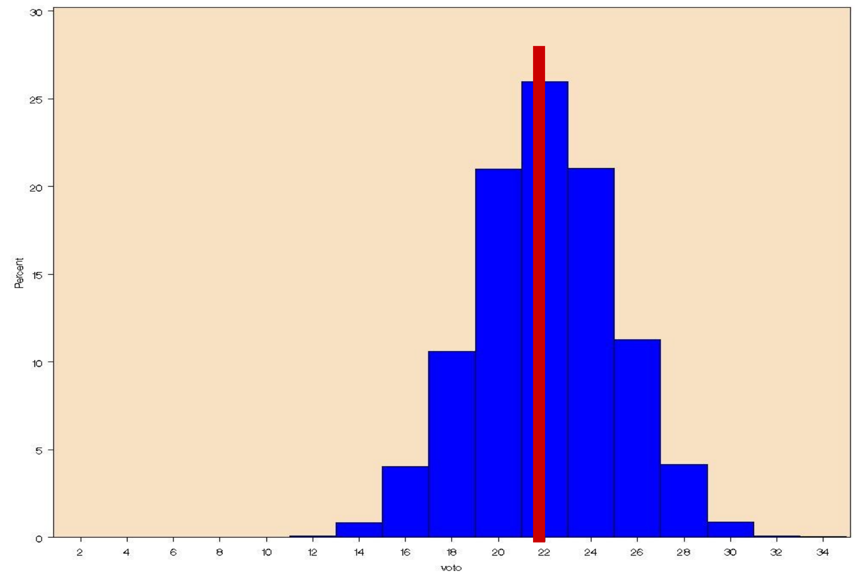
GLM Assumptions

$$y_i = a + e_i$$



Errors are independent (not related to each other)

GLM Assumptions



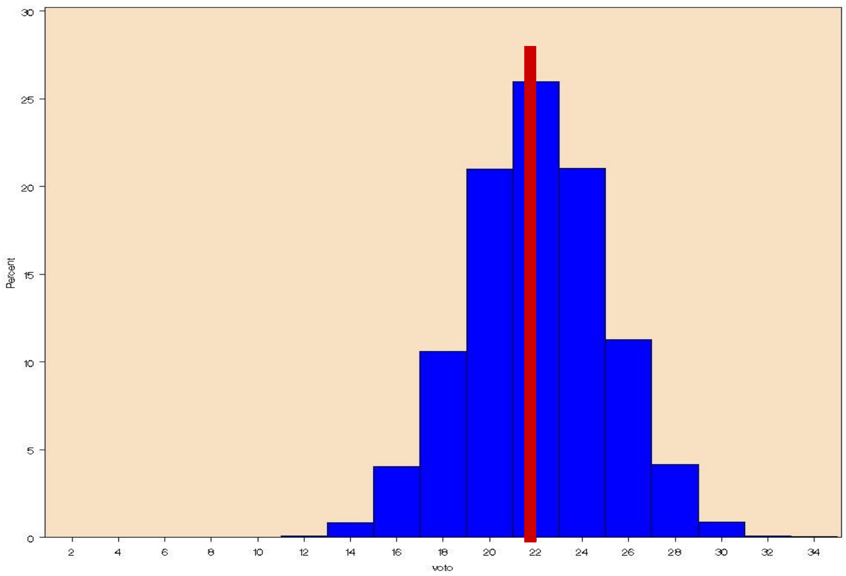
The estimated value is a **fixed** parameter

$$y_i = a + e_i$$

$$corr(e_i, e_j) = 0$$

Random variations are uncorrelated

GLM Assumptions



The estimated value is a **fixed** parameter

$$y_i = a + e_i$$

$$corr(e_i, e_j) = 0$$

Random variations are uncorrelated

GLM

When the assumptions are NOT met because the data, and thus the errors have more complex structures, we generalize the GLM to the Linear Mixed Model

Linear Mixed Model

GLM

- Regression
- T-test
- ANOVA
- ANCOVA
- Moderation
- Mediation
- Path Analysis

LMM

- Random coefficients models
- Random intercept regression models
- One-way ANOVA with random effects
- One-way ANCOVA with random effects
- Intercepts-and-slopes-as-outcomes models
- Multi-level models

Software

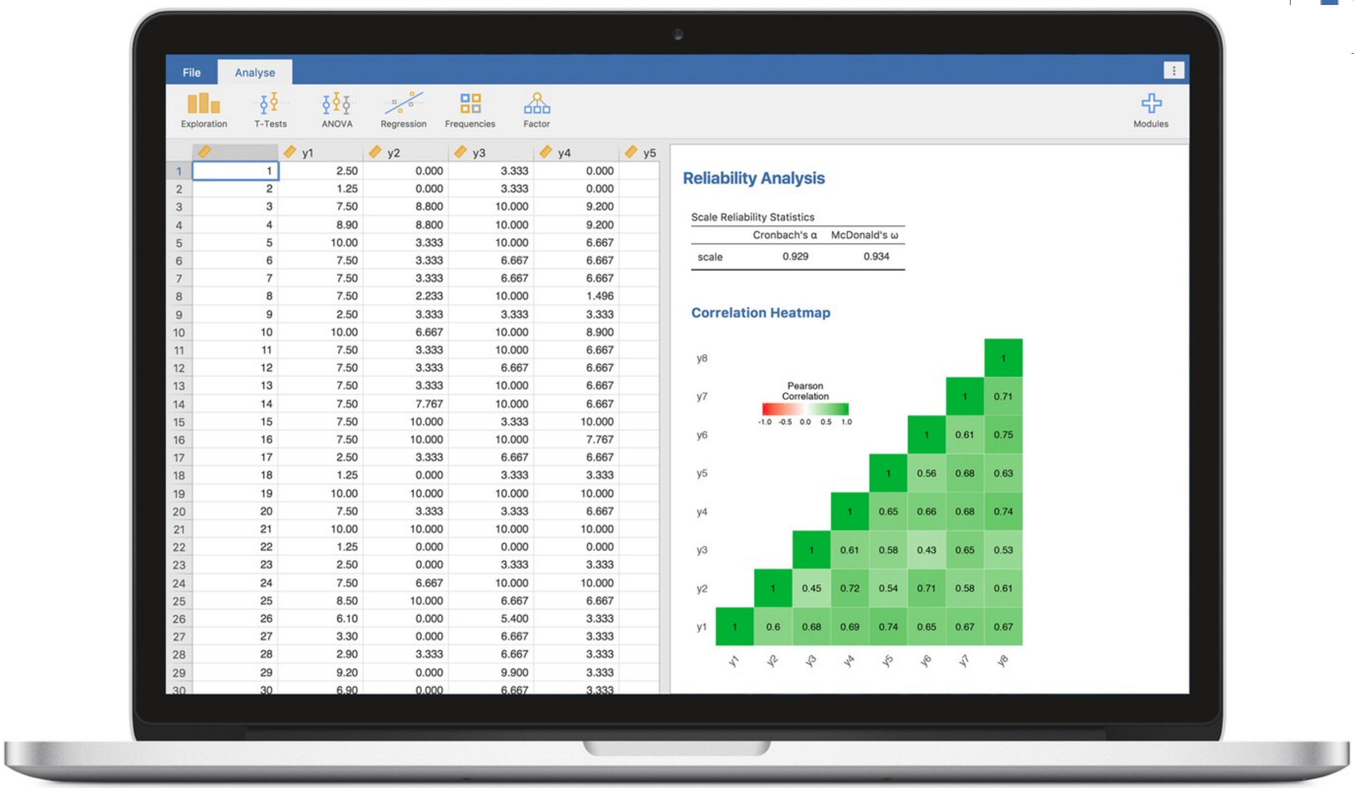
SPSS

R

jamovi Stats. Open. Now.

Jamovi

www.jamovi.org



jamovi Stats. Open. Now.

Example "beers"

Let's consider the case where the beer-smile research was conducted by gathering data in several different bars

For each participant we measured # of beers and # of smiles

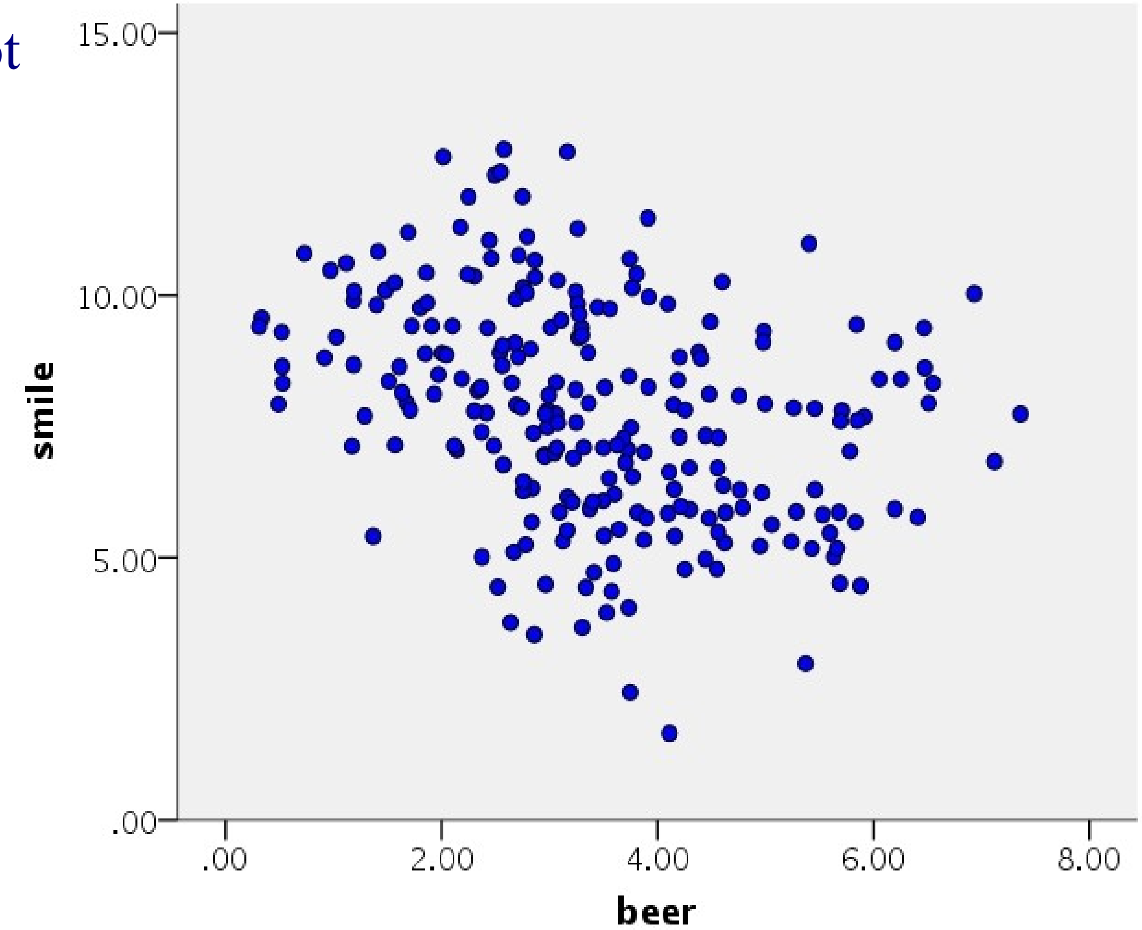
bar

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	a	3	1.3	1.3	1.3
	b	14	6.0	6.0	7.3
	c	22	9.4	9.4	16.7
	d	21	9.0	9.0	25.6
	e	14	6.0	6.0	31.6
	f	20	8.5	8.5	40.2
	g	24	10.3	10.3	50.4
	h	12	5.1	5.1	55.6
	i	16	6.8	6.8	62.4
	l	22	9.4	9.4	71.8
	m	21	9.0	9.0	80.8
	n	15	6.4	6.4	87.2
	o	16	6.8	6.8	94.0
	p	11	4.7	4.7	98.7
	q	3	1.3	1.3	100.0
	Total	234	100.0	100.0	

For a total of 234 participants

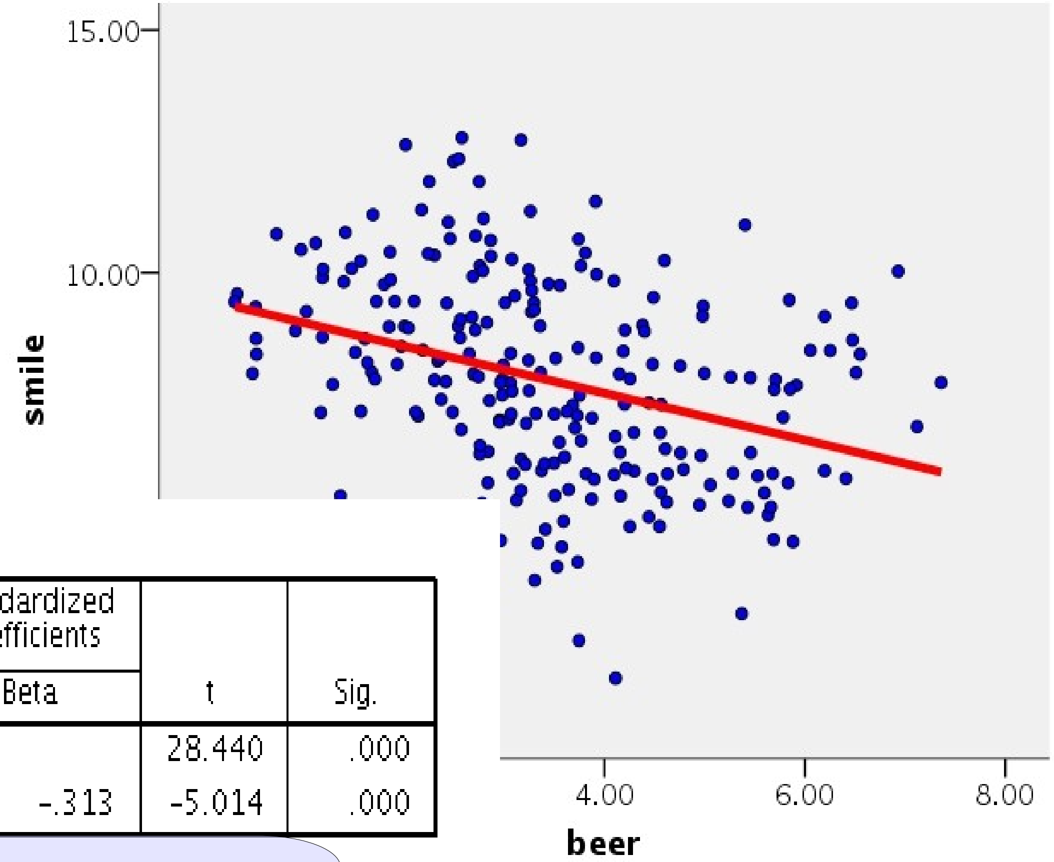
Example “beers” 2

As compared with the example with a few participants, now we have a very different scatterplot



Example “beers” 2

A simple regression confirms that results are indeed different



Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	9.303	.327		28.440	.000
	beer	-.432	.086	-.313	-5.014	.000

a. Dependent Variable: smile

Negative effect

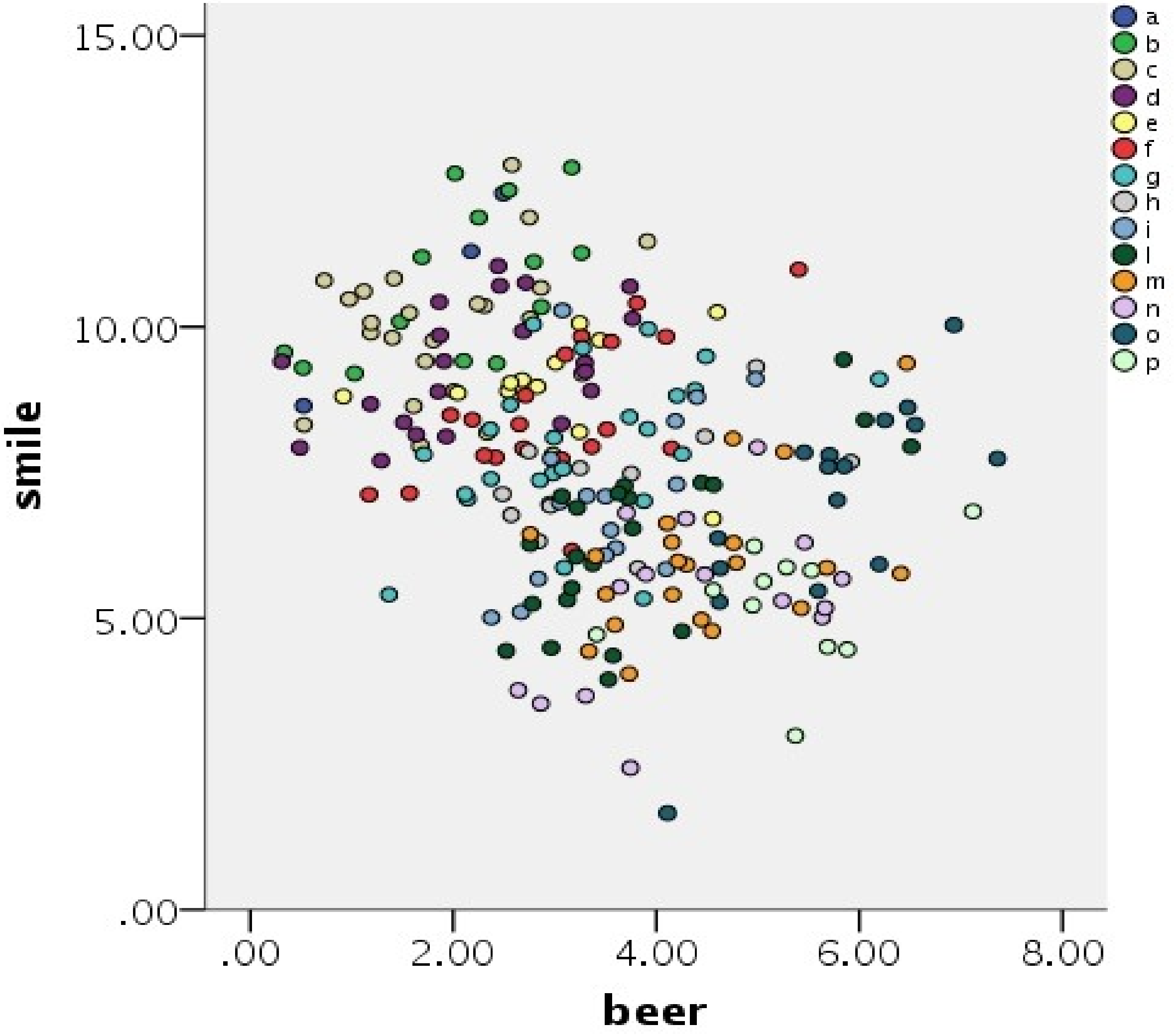
Why

Results may be biased by a mis-specification of the model, where the structure of the data is not taken into account

- In fact:
  - Subjects are sampled in clusters specified by **bars**
  - Each bar may have specific characteristics (quality, entertainment, etc) that may affect the measured variables
  - Subjects within the same bar may be more similar than across bars

Scatterplot by Bar

Let's see the data broken down by bar

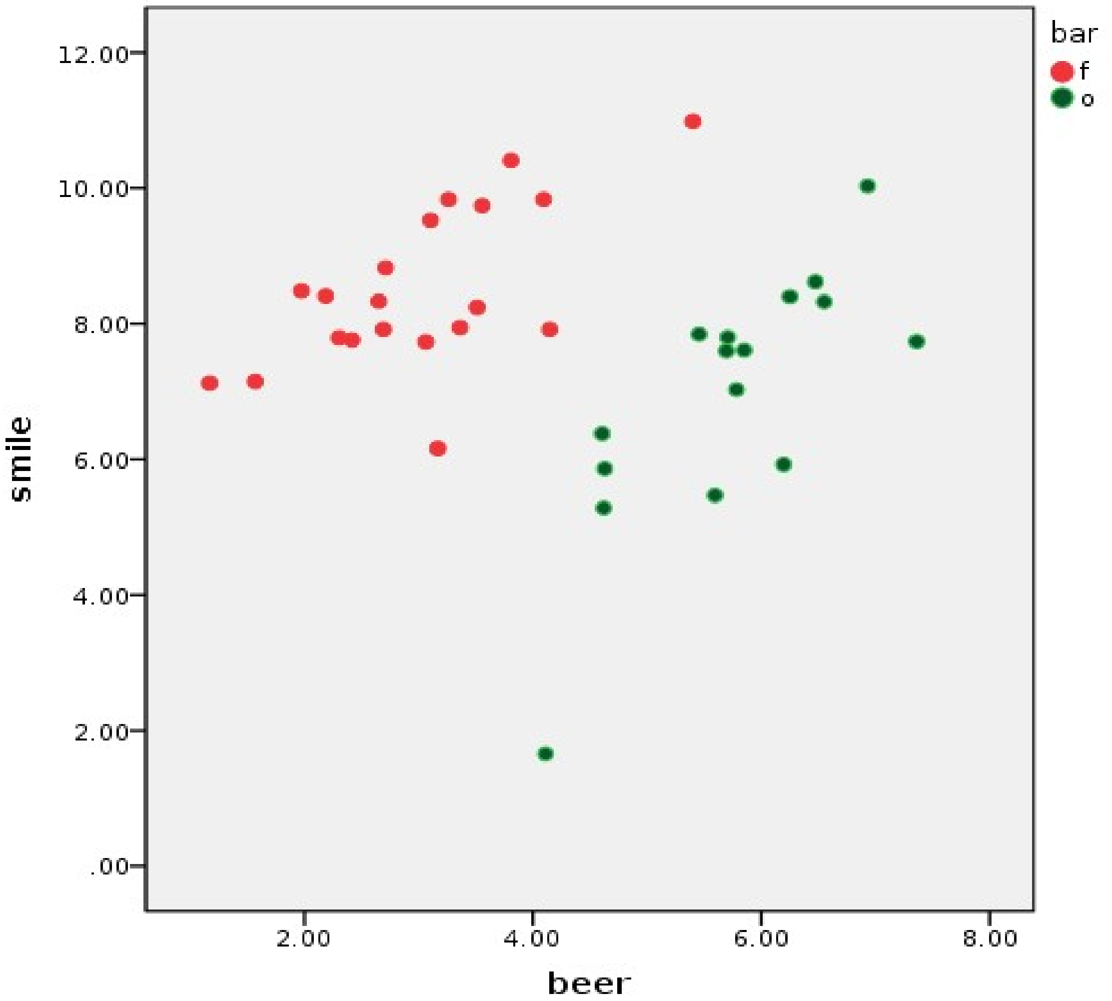


Bar

Scatterplot by Bar

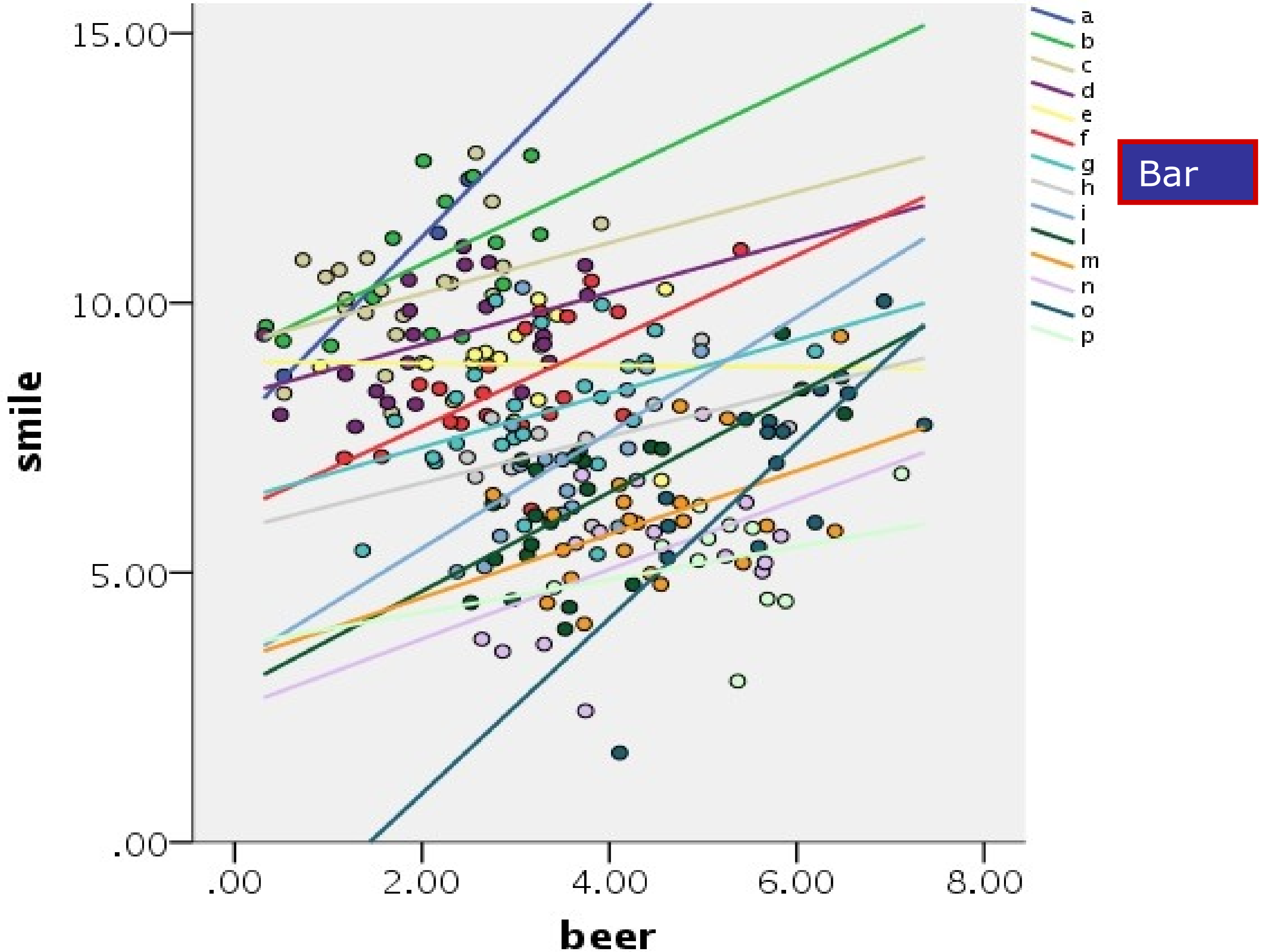
Let's see the data only for bar “f” and “o”

Bar



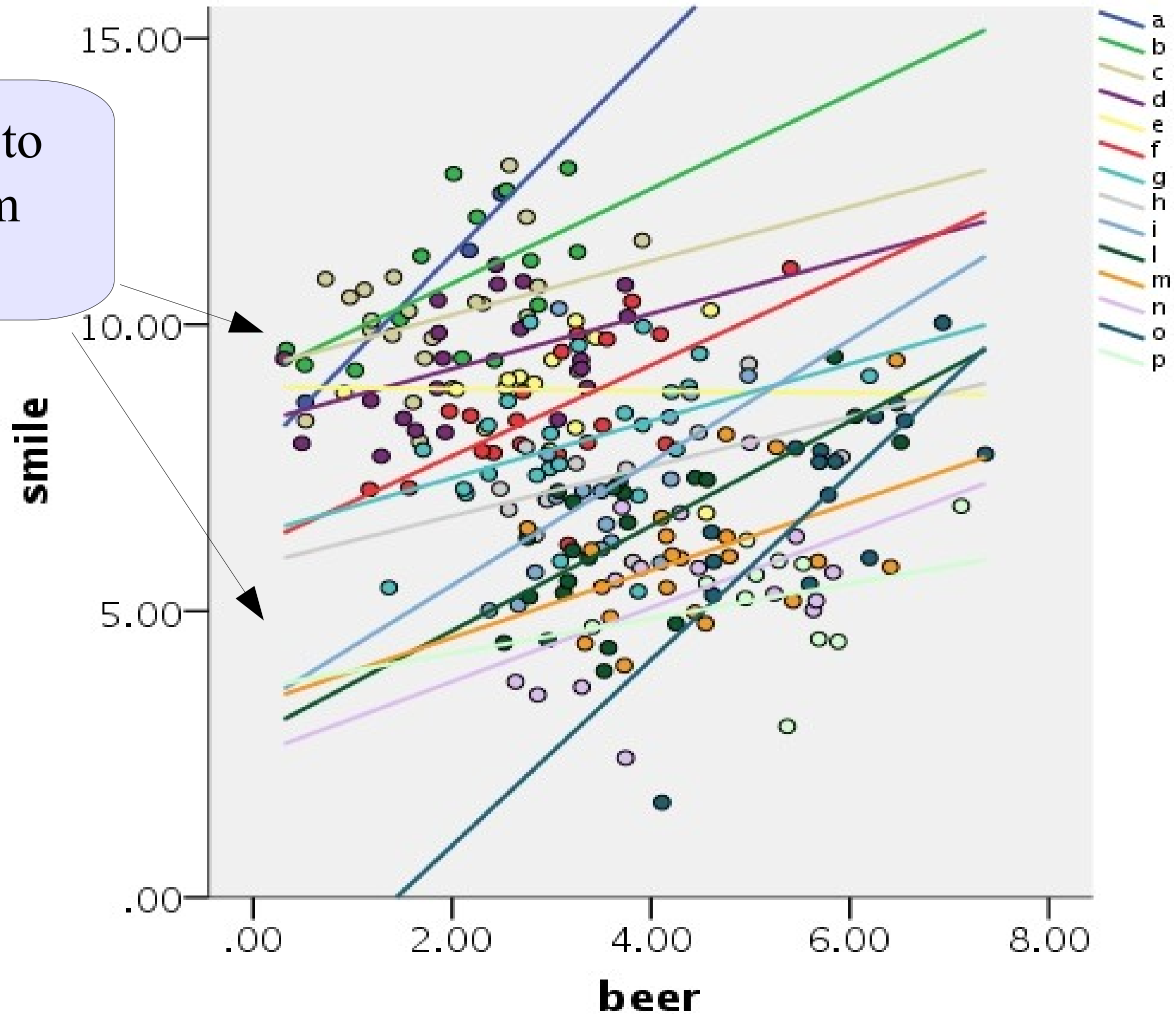
Scatterplot by Bar

It seems that the relations between IV and DV is positive, but within each bar



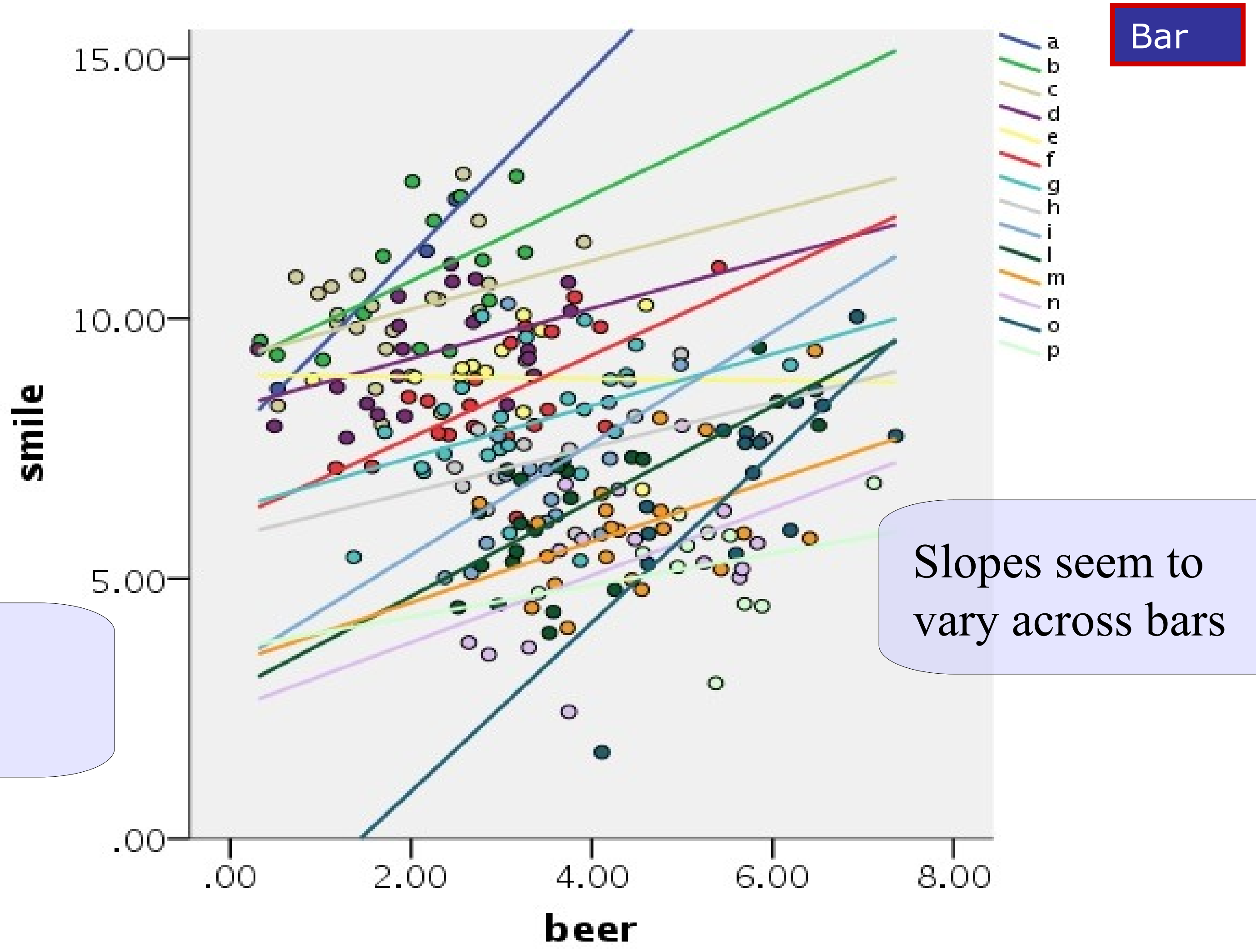
Scatterplot by Bar

Intercepts seem to be different from bar to bar



Bar

Scatterplot per Bar



Slopes are all positive

Slopes seem to vary across bars

The Model

- It seems that considering the participants as all equivalent and independent one each other (GLM assumption) does not fit our data

- It seems that a better model should allow each bar (each cluster) to have a different regression line (a different intercept and **b** coefficient)

The Model

- Let's define a model with a regression line for each cluster

y_{ij} Smiles of subject i in the cluster j

$$\hat{y}_{ia} = a_a + b_a \cdot x_{ia}$$

$$\hat{y}_{ib} = a_b + b_b \cdot x_{ib}$$

$$\hat{y}_{ic} = a_c + b_c \cdot x_{ic}$$

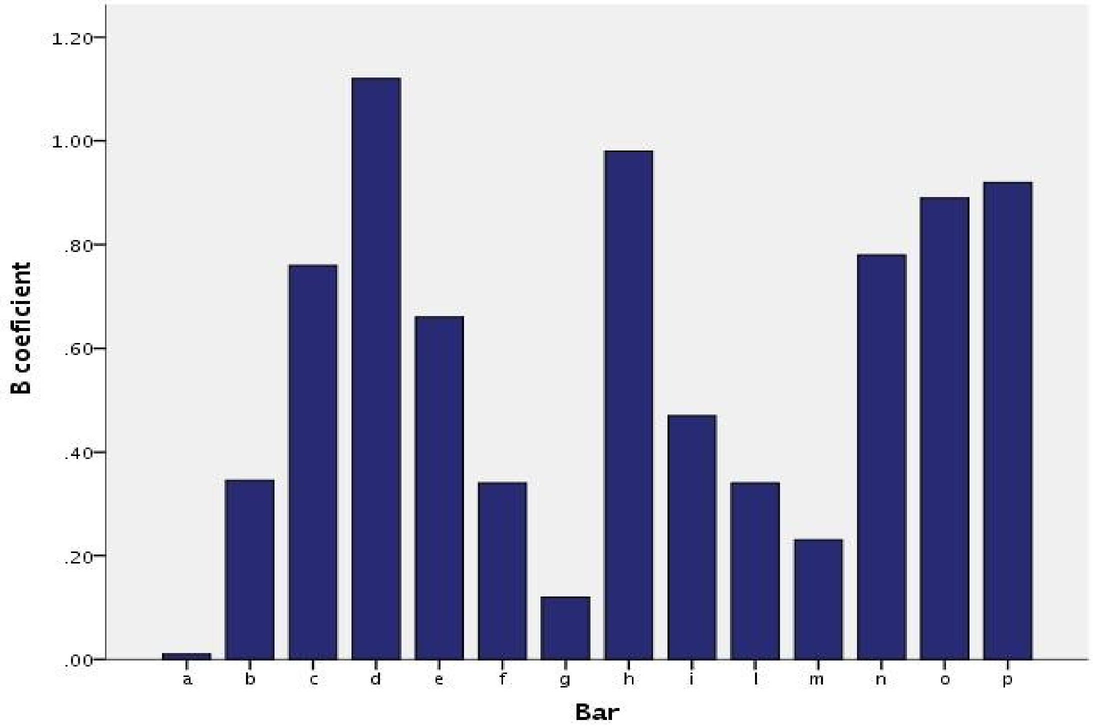
$$\hat{y}_{ij} = a_j + b_j \cdot x_{ij}$$

In these regressions the coefficients may vary from cluster to cluster: **they are not Fixed**

Varying coefficients

- If coefficients may vary, they will have a distribution

A possible distribution of coefficients **b** estimated for different clusters

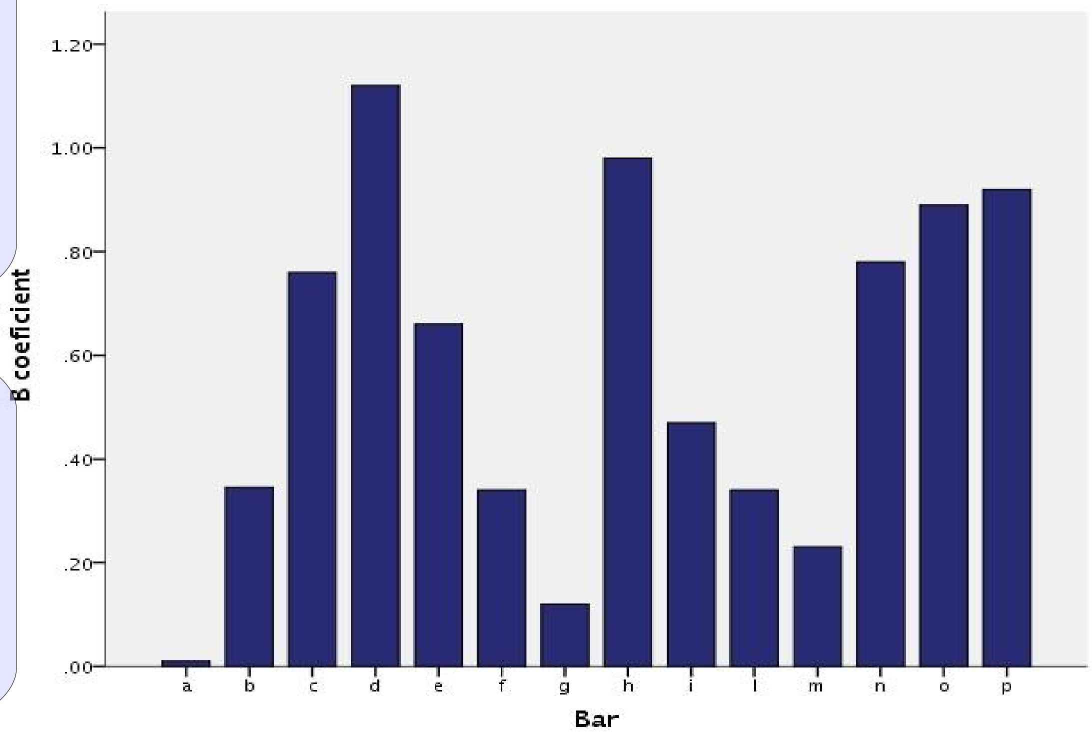


Random coefficients

- Varying coefficients are called random coefficients

Coefficients will exhibit variability

That is: in the **population** there exist different coefficients, a sample of which we estimated using the clustered data

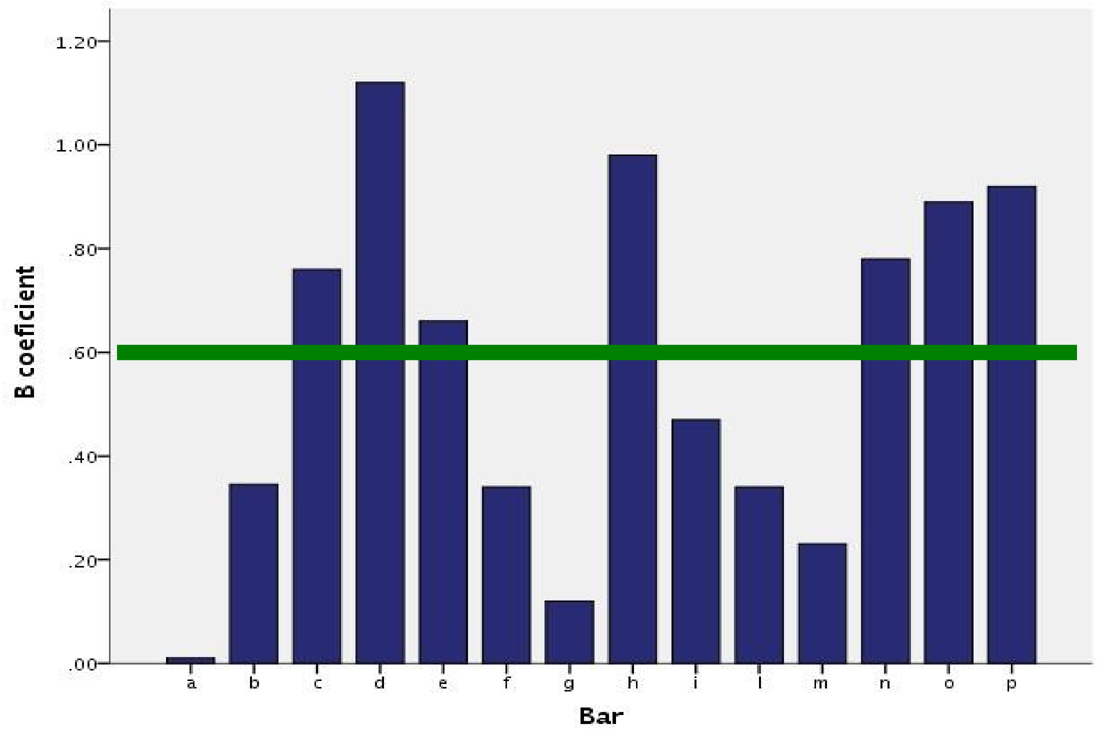


Average of the coefficients

- If coefficients vary as a variable in the population, they will have a mean and a variance, that we can estimate in our data

Mean

$$\bar{b} = \frac{\sum_j b_j}{k}$$



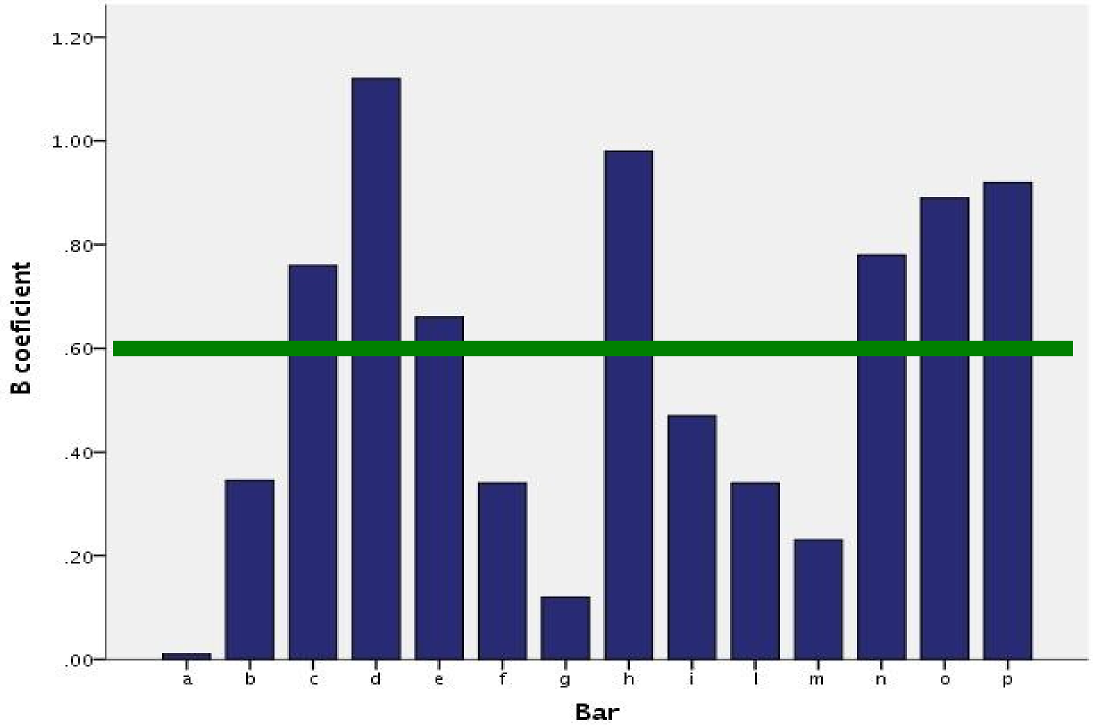
Fixed coefficients

- If coefficients vary as a variable in the population, they will have a mean and a variance, that we can estimate in our data

Mean

$$\bar{b} = \frac{\sum_j b_j}{k}$$

Recall the mean is a fixed parameter for a distribution, and so is the mean of the coefficients: it is a **fixed effect**



The Model

- We can now define a model with a regression for each cluster and the mean values of coefficients

One regression per cluster

$$\hat{y}_{ij} = a_j + b_j \cdot x_{ij}$$

Each coefficient is defined as the deviation from the mean coefficient

$$b'_j = b_j - \bar{b}$$

Overall model

$$\hat{y}_{ij} = a_j + b'_j \cdot x_{ij} + \bar{b} \cdot x_{ij}$$

The Model

- We can now define a model with a regression for each cluster and the mean value of coefficients

Overall model

$$\hat{y}_{ij} = a_j + b'_j \cdot x_{ij} + \bar{b} \cdot x_{ij}$$

Random coefficients

Fixed coefficient

The mixed model

- The same goes for the intercepts

One regression per cluster

$$\hat{y}_{ij} = a_j + b_j \cdot x_{ij}$$

Intercepts as deviations from the average intercept

$$a'_j = a_j - \bar{a}$$

Overall model

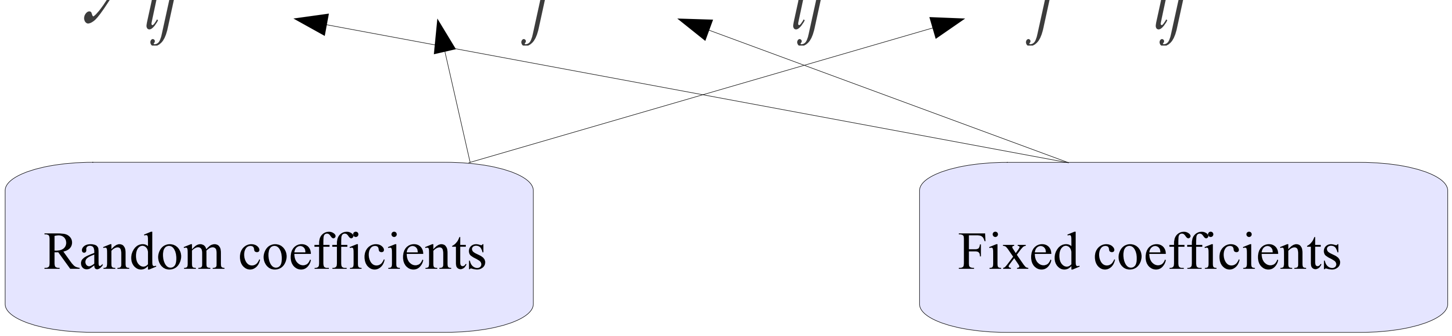
$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

The mixed model

- We can now define a model with a regression for each cluster and the mean values of coefficients

Overall model

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$



A GLM which contains both fixed and random effects is called a Linear Mixed Model

The mixed model

- Interpretation

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

The DV score (smiles) for participant i in the cluster (bars) j is a function of ...

The mixed model

Interpretation: y_ij is a function of...

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

The average of the expected value for x=0, across clusters

That is: For x=0, how big is y on average

The mixed model

Interpretation: y_ij is a function of...

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

Each expected value of y for x=0, in each cluster (bar) as a deviation from the overall intercept

For x=0, how much to add or subtract to the average intercept for that particular cluster

The mixed model

Interpretation: y_ij is a function of...

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

The average effect of x on y, averaged across clusters

On average, the expected change in the DV for a unit change in the IV

The mixed model

Interpretation: y_ij is a function of...

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

The deviation from the average effect of x for cluster j

How much the effect of x increases (or decreases) in that particular cluster

GLM as a special case

It is clear that everything we know for the GLM applies here: the GLM is in fact a special case of the LMM, where there are not random effects

LMM

$$\hat{y}_{ij} = \bar{a} + a'_j + \bar{b} \cdot x_{ij} + b'_j \cdot x_{ij}$$

GLM

$$\hat{y}_{ij} = \bar{a} + \bar{b} \cdot x_{ij}$$

Notation

For clarity, it's better to use the following notation

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + b_j \cdot x_{ij} + e_{ij}$$

y_{ij}, x_{ij}	**Scores for case i in cluster j**
$\bar{a}, \bar{b}$	**Fixed effects**
a_j, b_j	**Effetti random as deviation from their mean**
e_{ij}	**Error associated to the case i**

Variance

For clarity, it's better to use the following notation

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + b_j \cdot x_{ij} + e_{ij}$$

σ_a	**Coefficients a variance**
σ_b	**Coefficients b variance**
σ	**Error variance (residual)**
σ_{ab}	**Covariance between a and b coefficients**

The mixed model

- In practice, mixed models allow to estimate the kind of effects we can estimate with the GLM, but they allow the effects to vary across clusters.
- Effects that vary across clusters are called **random effects**
- Effects that do not vary (the ones that are the same across clusters) are said to be **fixed effects**

The mixed model

- To specify a correct model, we only need to understand if there are **clusters of cases** (measures or subjects) and decide which coefficients (intercepts or b coefficients) may vary across those clusters

- The fixed effects of the model are interpreted like in the GLM (regression/ANOVA)

- **Random effects** are generally not interpreted, but we can look at their variance to decide to keep them as random (variance>0) or fix them.

- In this way we take into the account the dependence among data

Building a model

To build a model in a simple way, we need to answer very few questions:

- What is (are) the cluster variable(s)?
- What are the fixed effects?
- What are the random effects?

A clustering variable

- **What is (are) the cluster variable(s)?**
- What are the fixed effects?
- What are the random effects?

  - Any variable that groups observations (cases or measurements) such that scores may be more similar within each group than across groups
  - Any variable whose levels (groups) are a sample of a larger population of levels (groups)
  - Example: bars created groups of scores (participants) that may be more similar within the bar that across bars

Fixed effects

- What is (are) the cluster variable(s)?
- **What are the fixed effects?**
- What are the random effects?

  - Any effect that we are interested in on average (as in a standard ANOVA/Regression)
  - Example: the effect of beer on smiles in general

Fixed effects

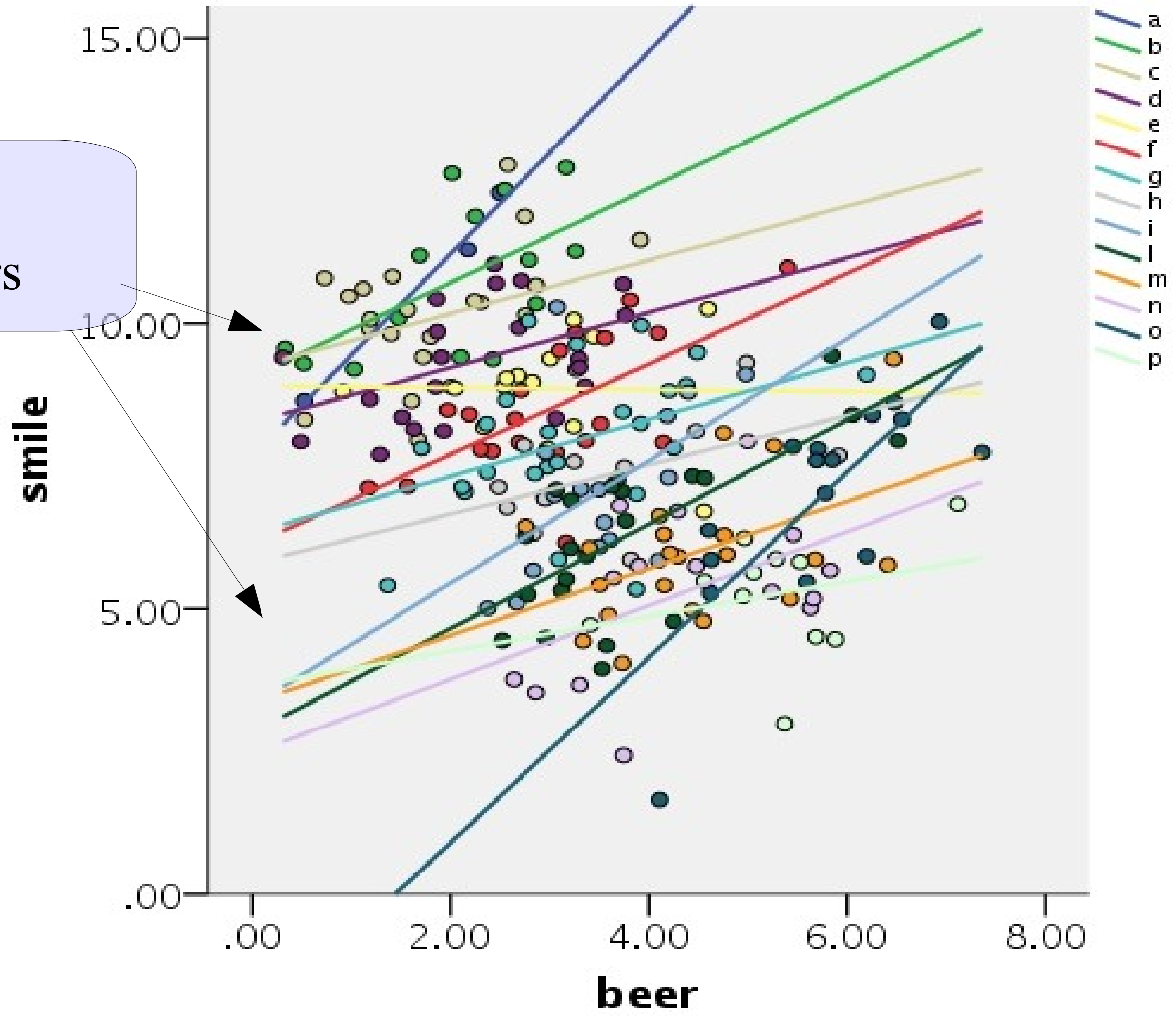
- What is (are) the cluster variable(s)?
- What are the fixed effects?
- **What are the random effects?**
  - Any effect that may vary from cluster to cluster
  - (Thus:) **Any effect that can be computed within each cluster**
  - Example: the intercepts and the effect of beer on smiles each bar

Beers at the bar

We start with a simple model

Bar

Intercepts may vary across bars



Beers at the bar

We define a model where each cluster is allow to have a different intercept, the rest of the model is like a standard regression

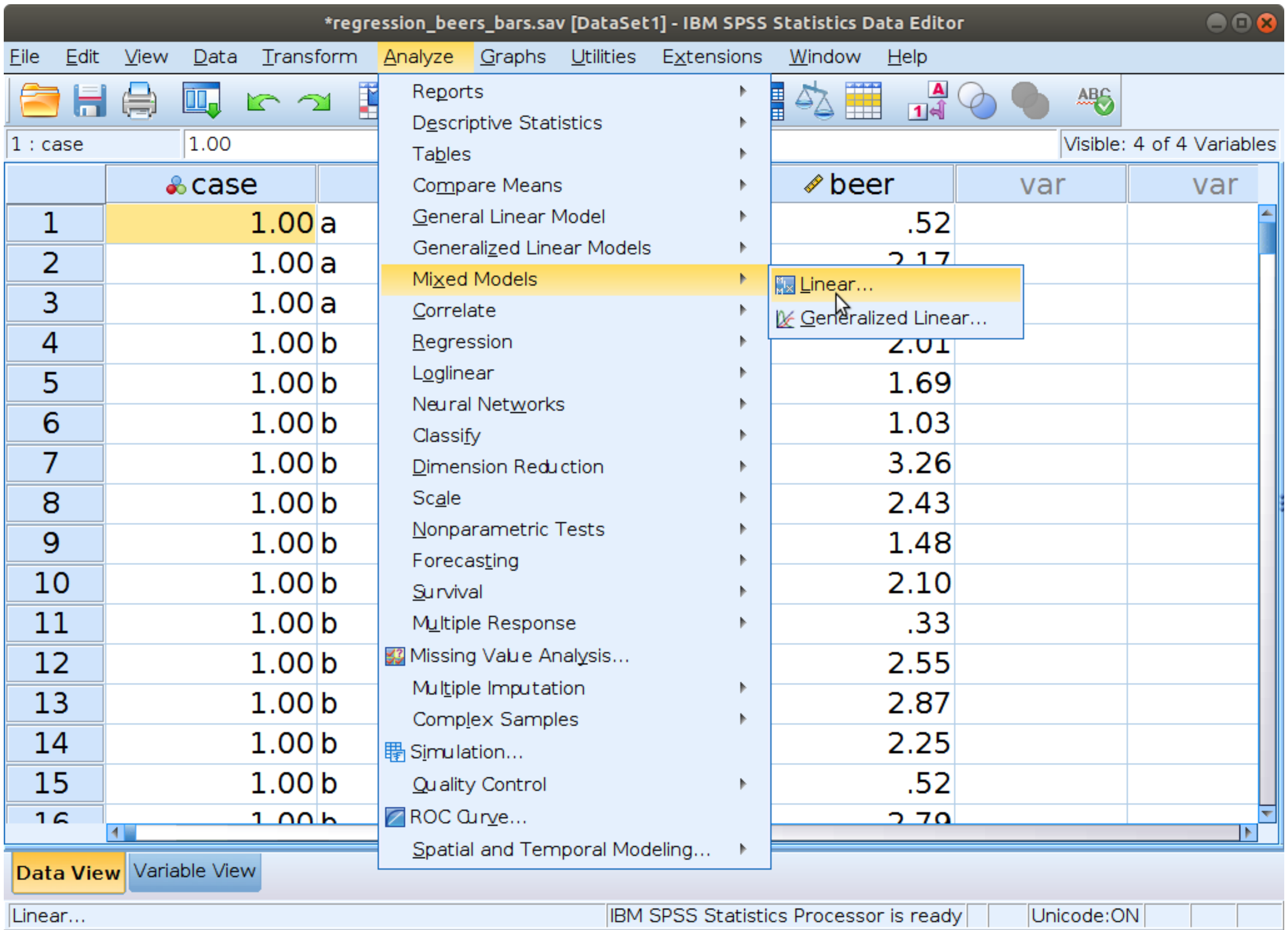
$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + e_{ij}$$

- Fixed effects? Intercept and beer effect
- Random effects? Intercepts
- Clusters? bar

Authors and books may call this model:
Random-intercepts regression
or
Intercepts-as-outcomes model

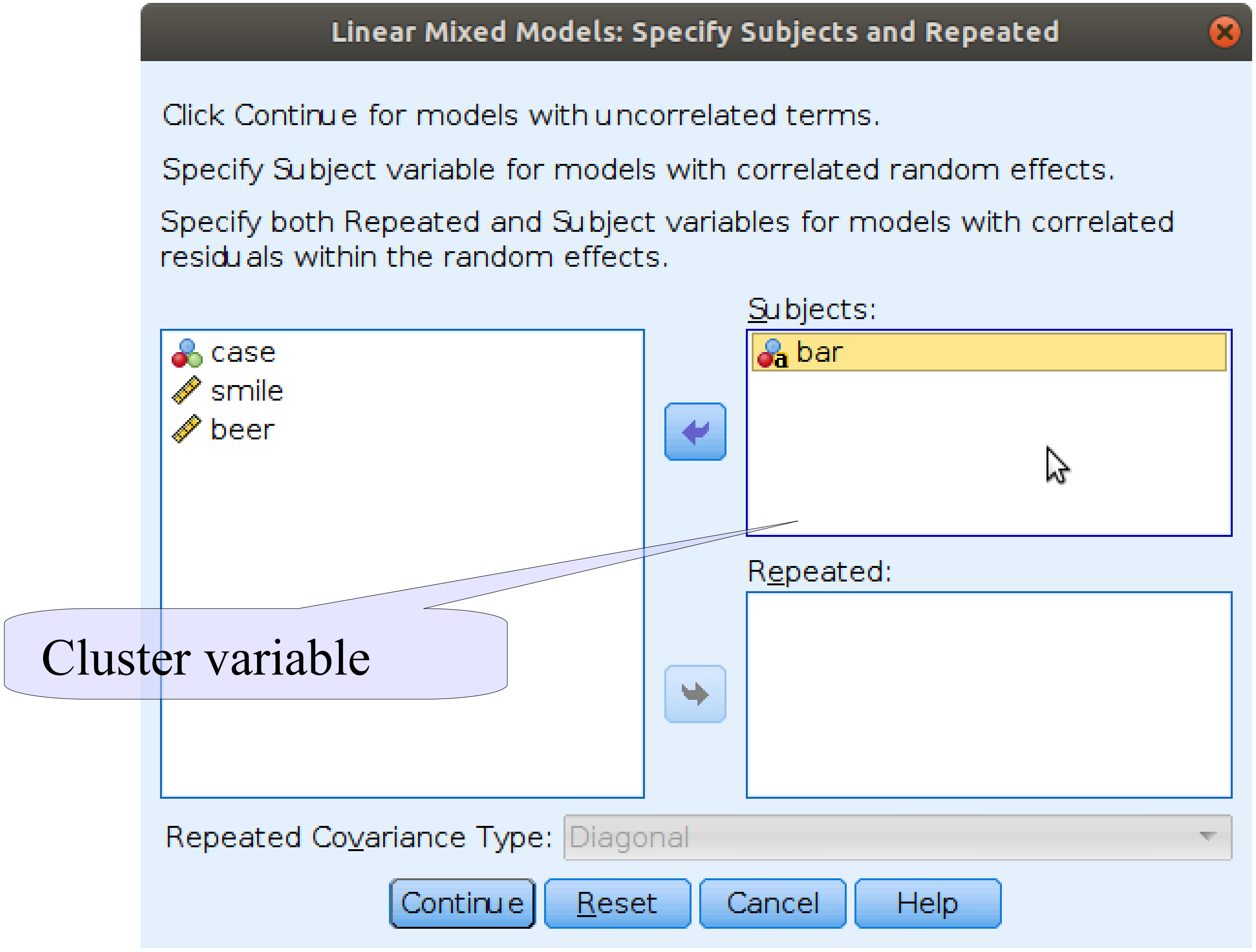
SPSS Input

Analyze → Mixed Models → Linear



SPSS Input

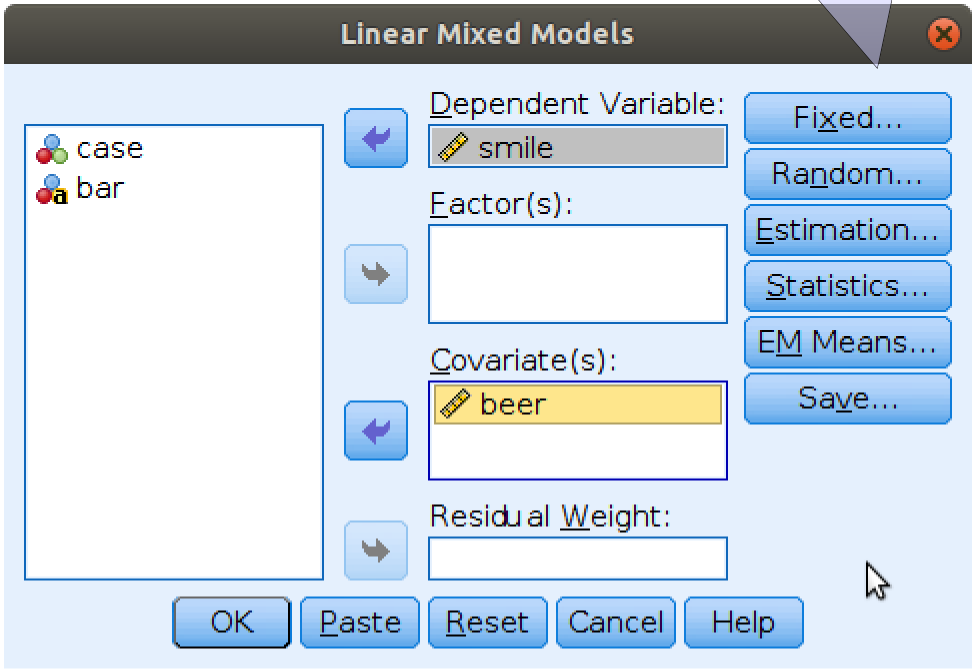
Analyze → Mixed Models → Linear



SPSS Input

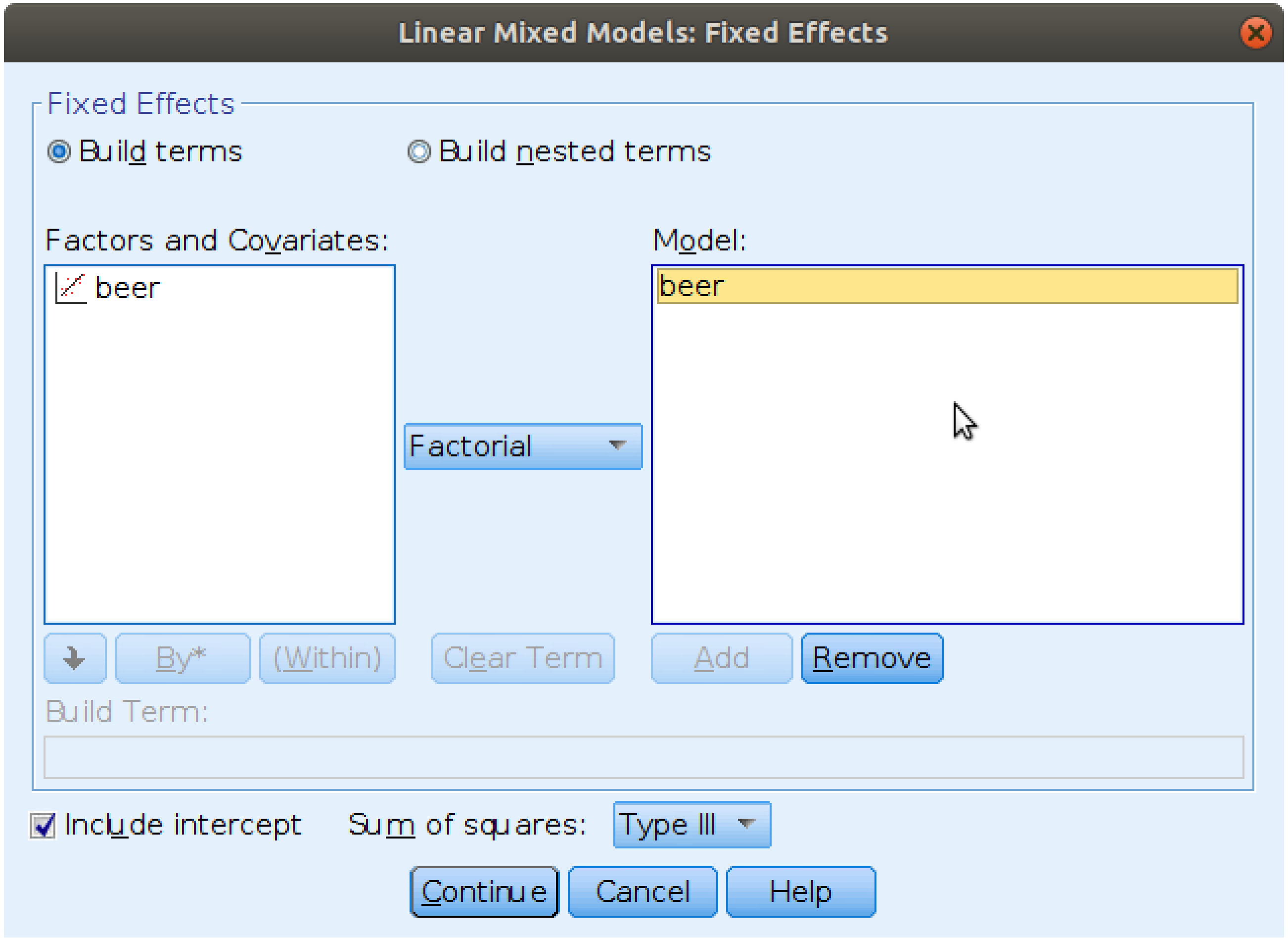
Analyze → Mixed Models → Linear

Then select Fixed



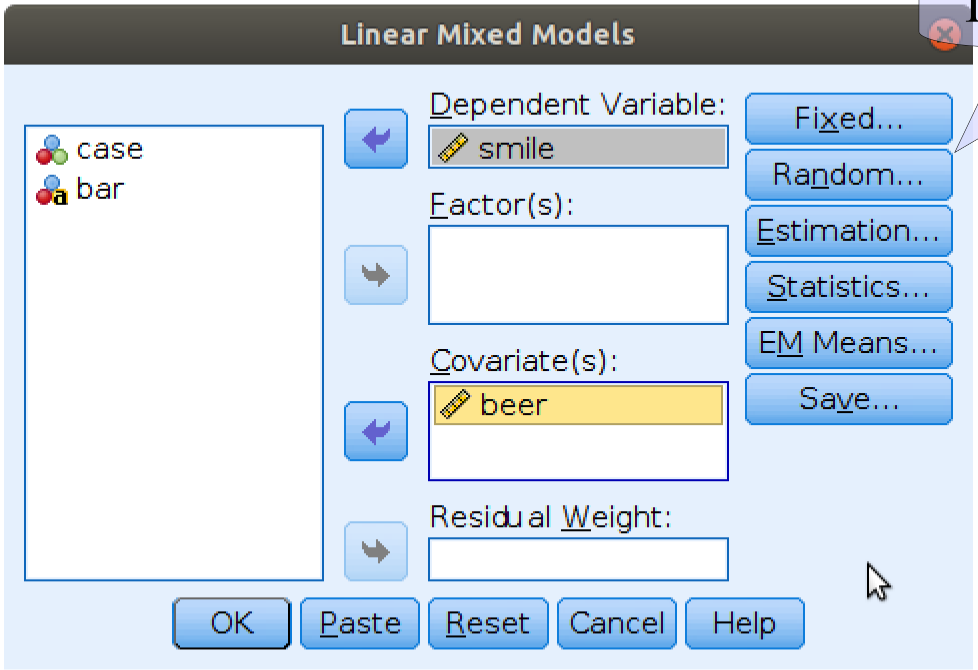
SPSS Input

Analyze → Mixed Models → Linear



SPSS Input

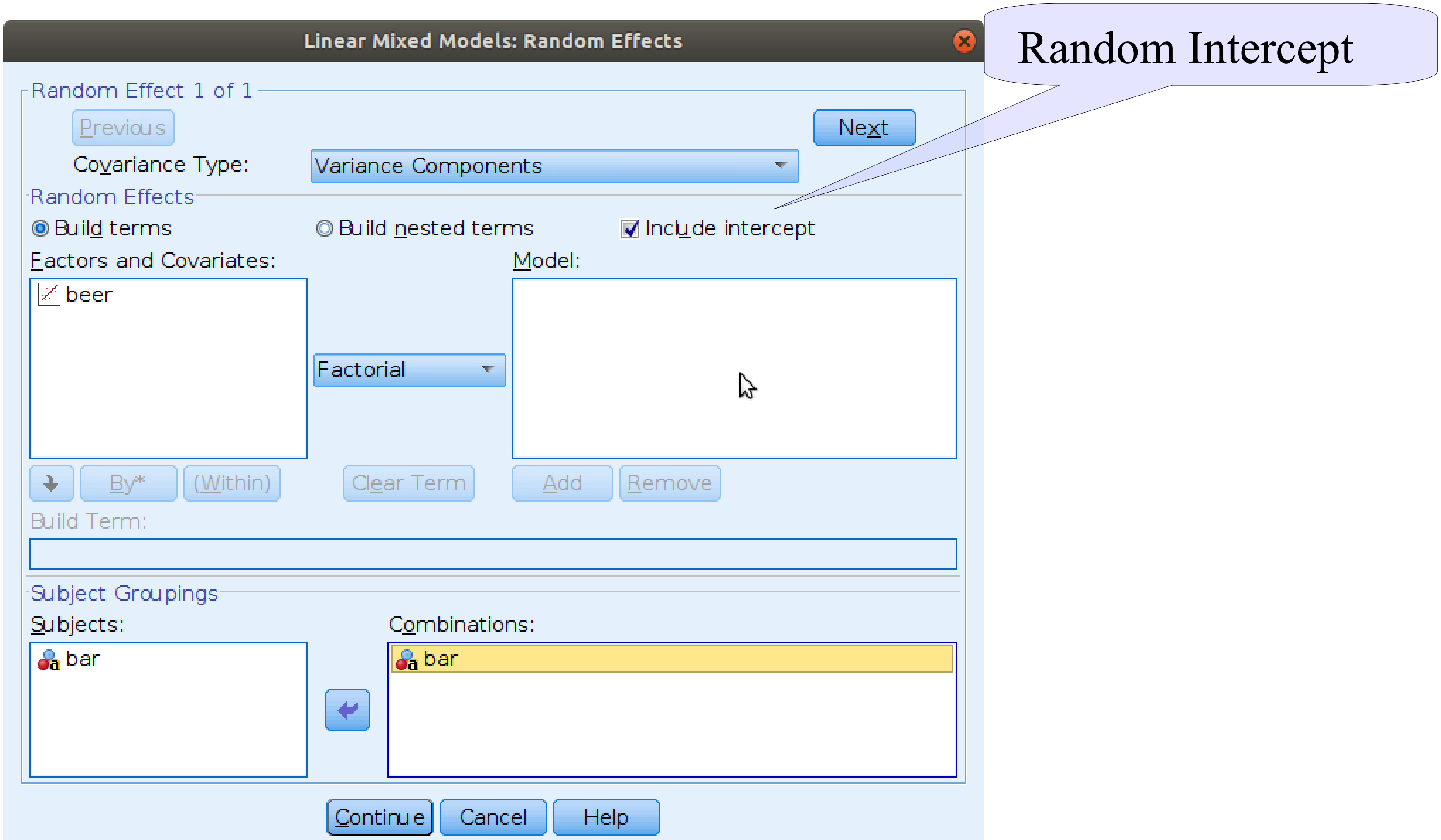
Analyze → Mixed Models → Linear



Then select Random

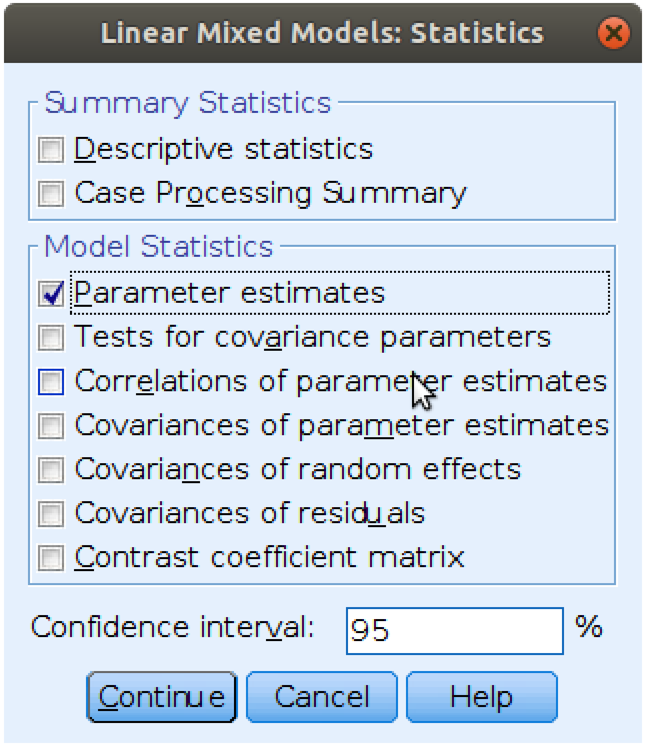
SPSS Input

Analyze → Mixed Models → Linear



SPSS Input

Analyze → Mixed Models → Linear



Print coefficients

SPSS syntax

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + e_{ij}$$

```
MIXED smile WITH beer by bar
  /CRITERIA=CIN(95) MXITER(100) MXSTEP(10) SCORING(1) SINGULAR(0.000000000001) HCONVERGE(0,
   ABSOLUTE) LCONVERGE(0, ABSOLUTE) PCONVERGE(0.000001, ABSOLUTE)
  /FIXED=beer| SSTYPE(3)
  /METHOD=REML
 /print solution TESTCOV
  /random intercept   | SUBJECT(bar) COVTYPE(un).
```

Fixed effects (intercept is included by default)

Random effects

Cluster variable

SPSS Output

Let's see if the model is how intended

Model Dimension[a]

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	beer	1		1	
Random Effects	Intercept	1	Identity	1	bar
Residual				1	
Total		3		4	

a. Dependent Variable: smile.

OK!

SPSS Output

We then check the variability of the random effects. If there is variability across bars, it means we were right to model the coefficients as random

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Residual		1.284725	.123256	10.423	.000	1.064501	1.550509
Intercept [subject = bar]	Variance	6.531614	2.584158	2.528	.011	3.007824	14.183668

a. Dependent Variable: smile.

Variance greater than 0

SPSS Output

If everything is fine, we interpret the fixed effects as in any other GLM (regression)

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	5.551071	.724190	18.295	7.665	.000	4.031359	7.070783
beer	.638704	.077690	227.901	8.221	.000	.485621	.791787

a. Dependent Variable: smile.

Intercept: On average, for zero beers we expect 5.5 smiles

SPSS Output

If everything is fine, we interpret the fixed effects as in any other GLM (regression)

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Intercept	5.551071	.724190	18.295	7.665	.000	4.031359	7.070783
beer	.638704	.077690	227.901	8.221	.000	.485621	.791787

a. Dependent Variable: smile.

B coefficient: On average,for each beer more, we expect .638 more smiles

R syntax

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + e_{ij}$$

random_beers.R*

Source on Save

```
setwd("/home/marcello/Skinner/Teaching/Phd/MIB")
library(lme4)
library(lmerTest)
library(foreign)
dat<-read.spss('data/regression_beers_bars.sav',to.data.frame = T)
head(dat)
mm1<-lmer(smile~1+beer+(1|bar),data=dat)
summary(mm1)
```

Load the required libraries

Random effects

Cluster variable

Fixed effects (intercept can be omitted as ti is included by default)

R Output

Let's see if the model is how it was intended

```
> summary(mm1)
Linear mixed model fit by REML ['merModLmerTest']
Formula: smile ~ 1 + (1 | bar) + beer
   Data: dat

REML criterion at convergence: 786.7
```

OK!

R Output

We then check the variability of the random effects. If there is variability across bars, it means we were right to model the coefficients as random

```
Random effects:
 Groups   Name        Variance Std.Dev.
 bar      (Intercept) 6.532    2.556
 Residual             1.285    1.133
Number of obs: 234, groups: bar, 15
```

Variance greater than 0

R Output

If you need a test for the random variances (Likelihood Ratio Test) run this:

```
dat<-read.spss('data/regression_beers_bars.sav',to.data.frame = T)
head(dat)
mm1<-lmer(smile~1+beer+(1|bar),data=dat)

rand(mm1)
```

```
> rand(mm1)
Analysis of Random effects Table:
    Chi.sq Chi.DF p.value
bar    201      1  <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
```

OK!

R Output

If everything is fine, we interpret the fixed effects as in any other GLM (regression)

```
Fixed effects:
             Estimate Std. Error        df t value Pr(>|t|)    
(Intercept)   5.55107    0.72419  18.29000   7.665 4.03e-07 ***
beer          0.63870    0.07769 227.88000   8.221 1.53e-14 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Intercept: On average, for zero beers we expect 5.5 smiles

Note that without library(lmerTest) you do not get the p.values!

R Output

If you prefer the F-test, use anova()

Default DF

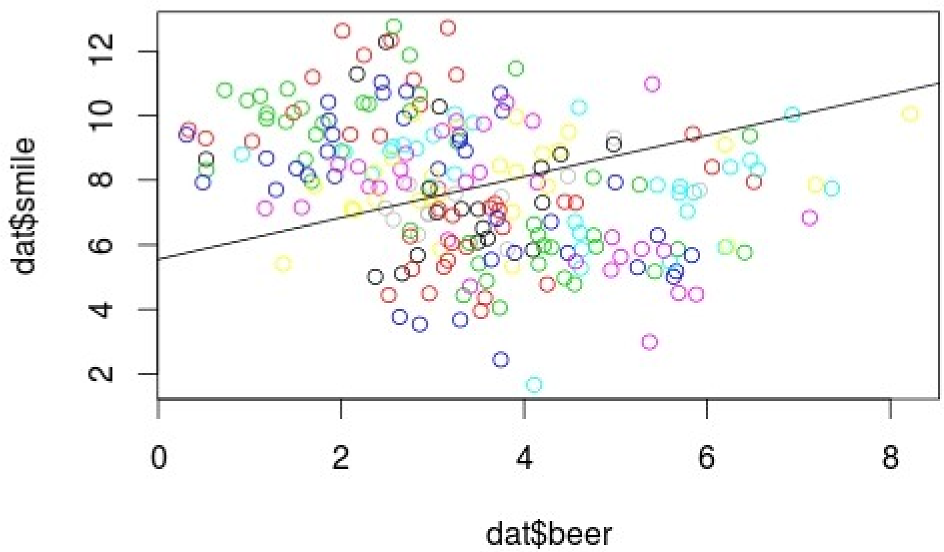
```
> anova(mm1)
Analysis of Variance Table of type 3  with  Satterthwaite
approximation for degrees of freedom
     Sum Sq Mean Sq NumDF  DenDF F.value    Pr(>F)
beer 86.831  86.831     1 227.88  67.588 1.532e-14 ***
---
```

R Output

Plot fixed effects:

```
summary(mm1)
plot(dat$smile~dat$beer,col=dat$bar)
abline(fixef(mm1))
```

fixef extracts fixed coefficients from the model

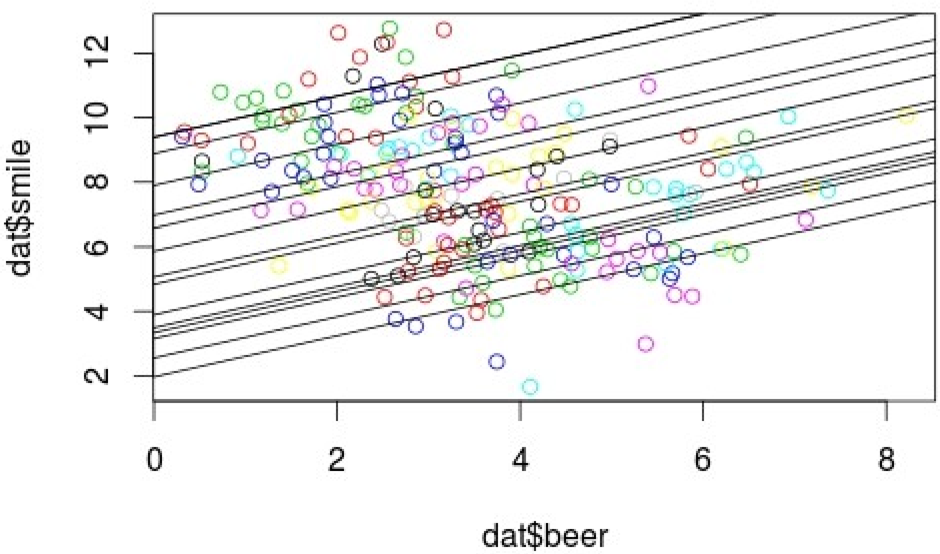


R Output

Plot random effects effects:

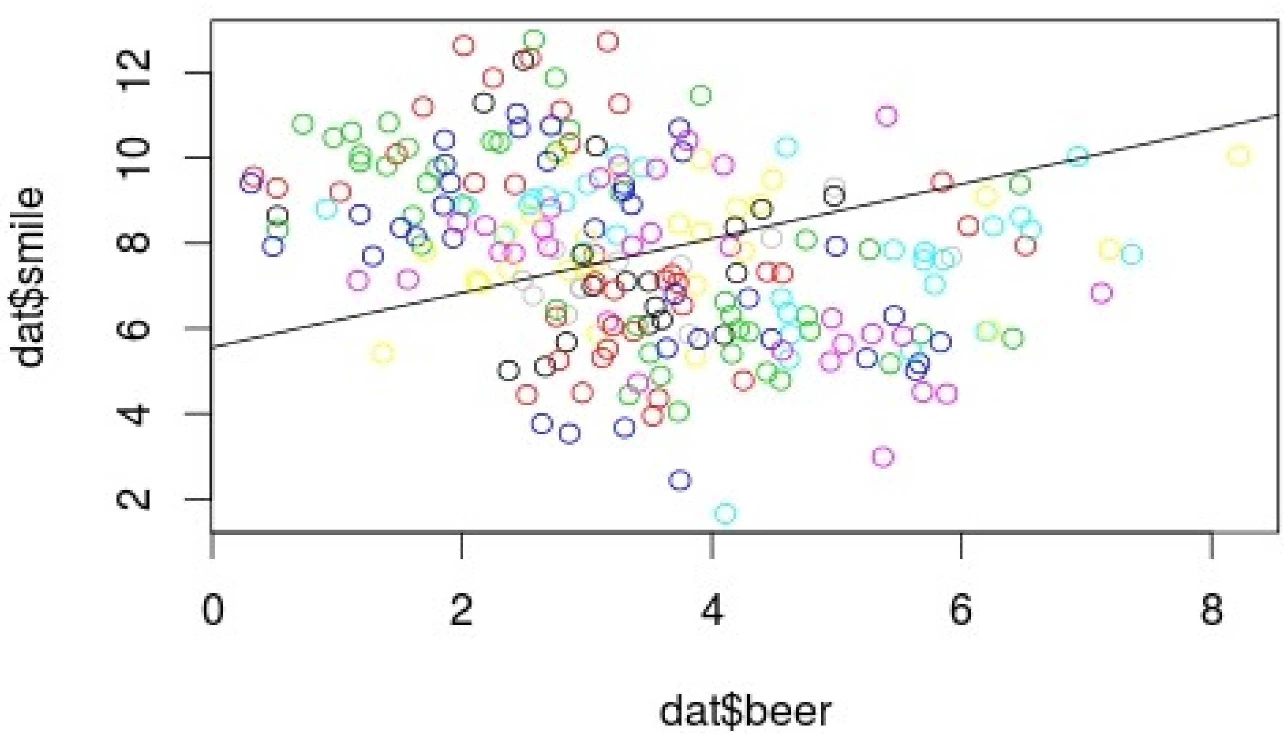
```
plot(dat$smile~dat$beer,col=dat$bar)
apply(coef(mm1)[[1]],1,abline)

```

coef extracts random coefficients from the model



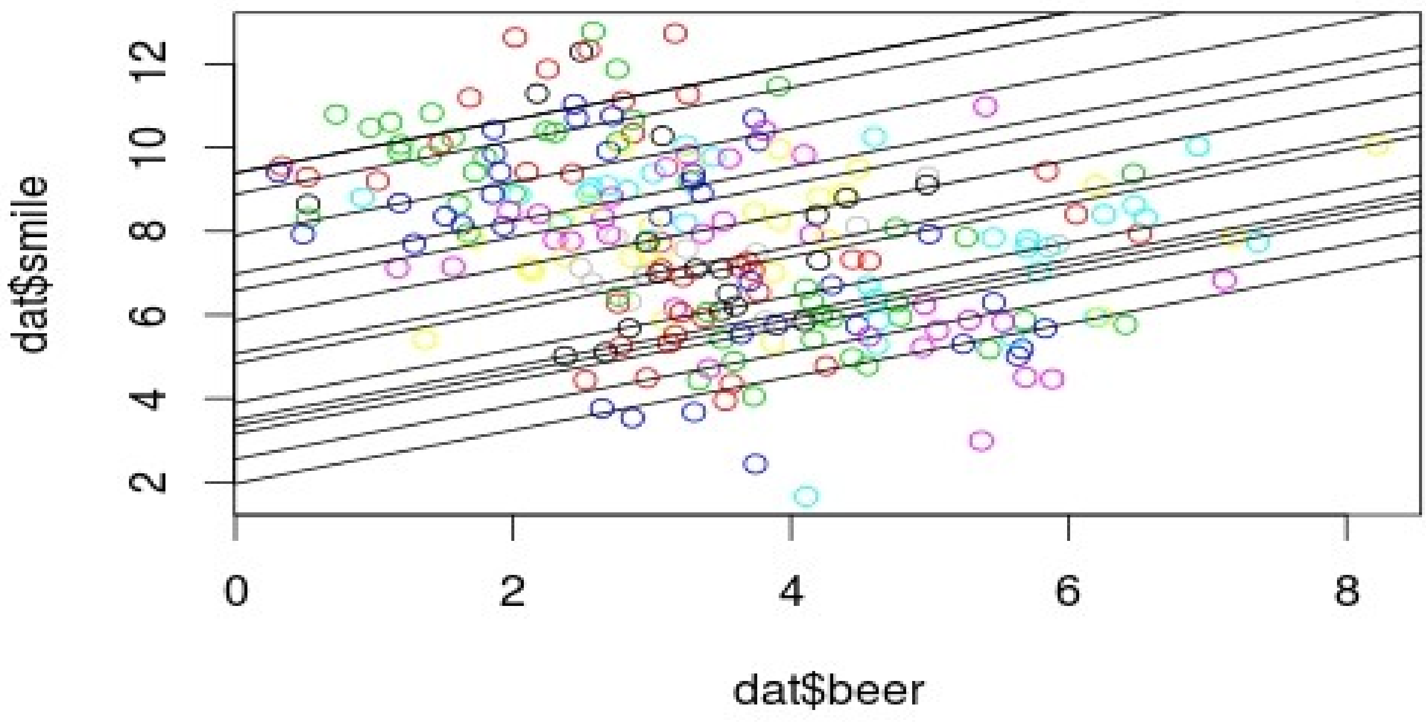
Plots

- Plotting fixed effects is simply plotting the straight line as in any linear model



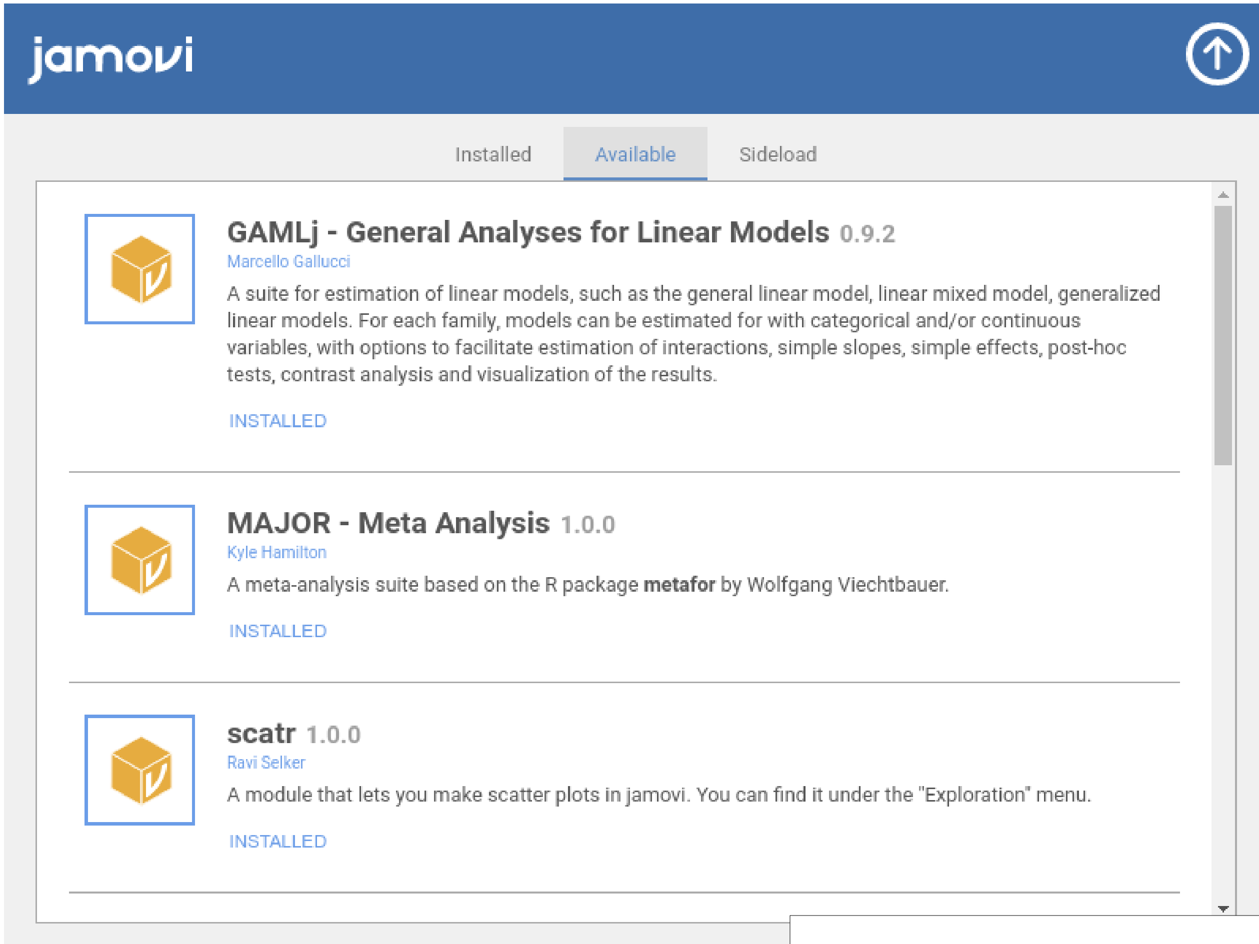
Plots

- Plotting random effects effects is plotting each specific sub-model of each cluster



Jamovi

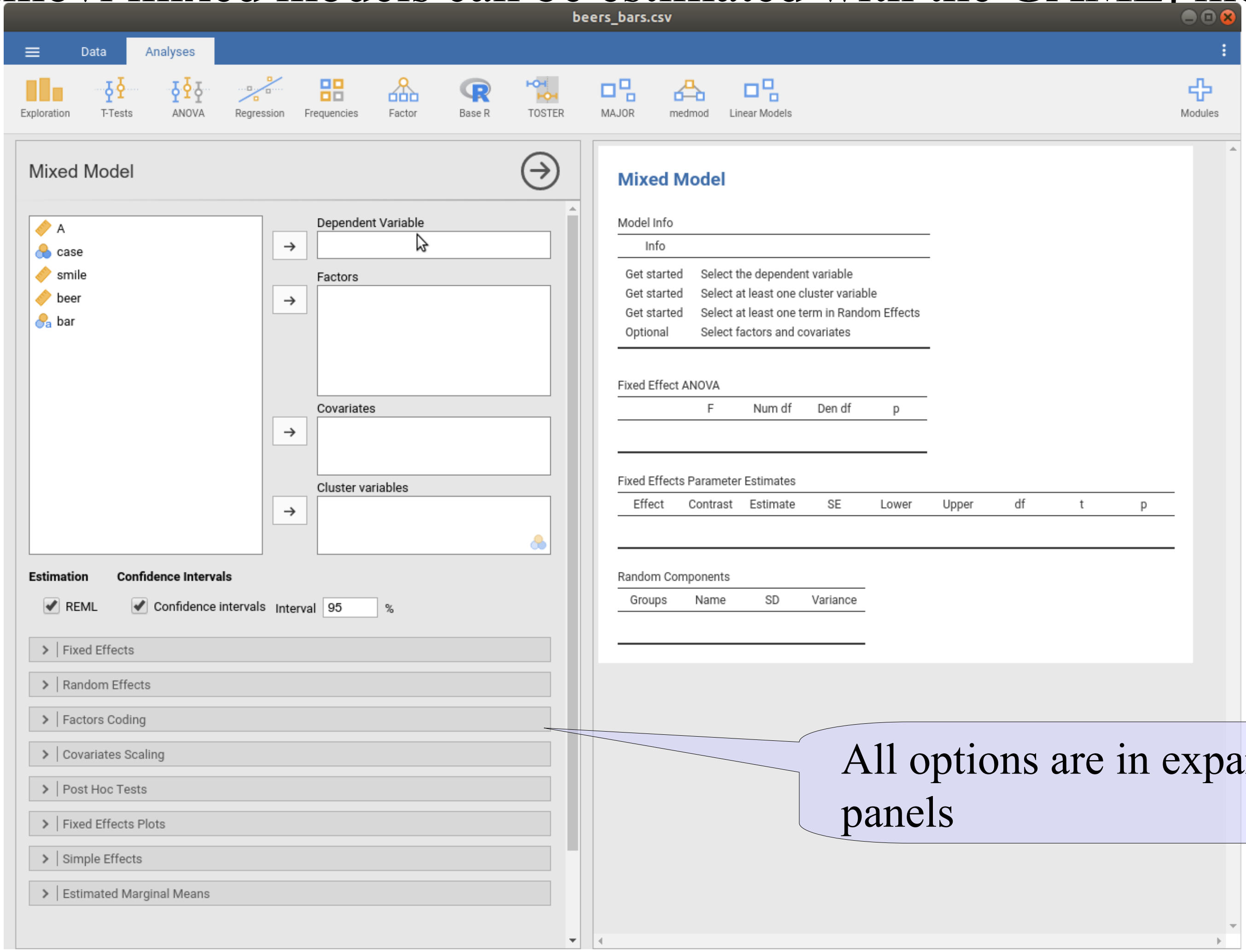
- In jamovi mixed models can be estimated with the GAMLj module



Docs and examples: https://mcfanda.github.io/gamlj_docs/

Jamovi

- In jamovi mixed models can be estimated with the GAMLj module

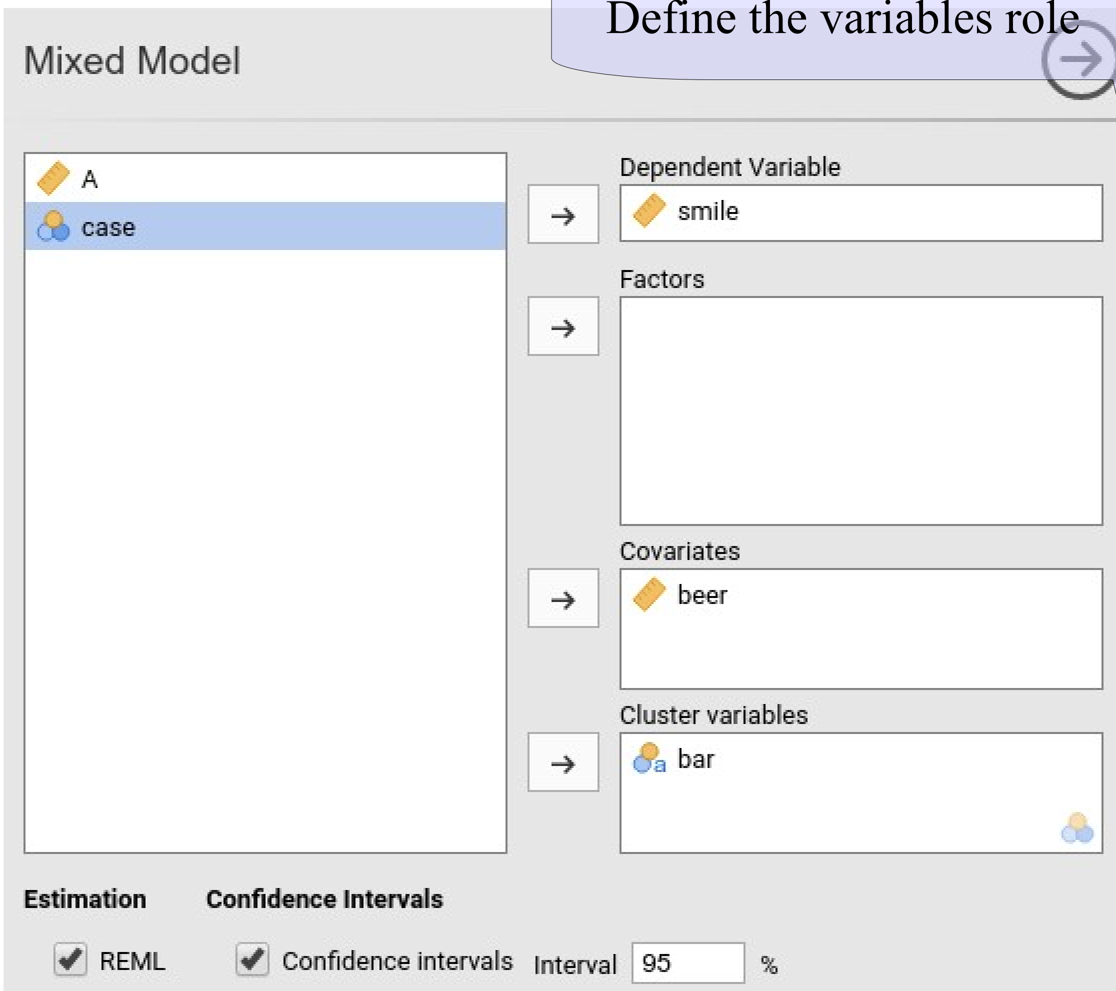


All options are in expandable panels

Jamovi

- In jamovi mixed models can be estimated with the GAMLj module

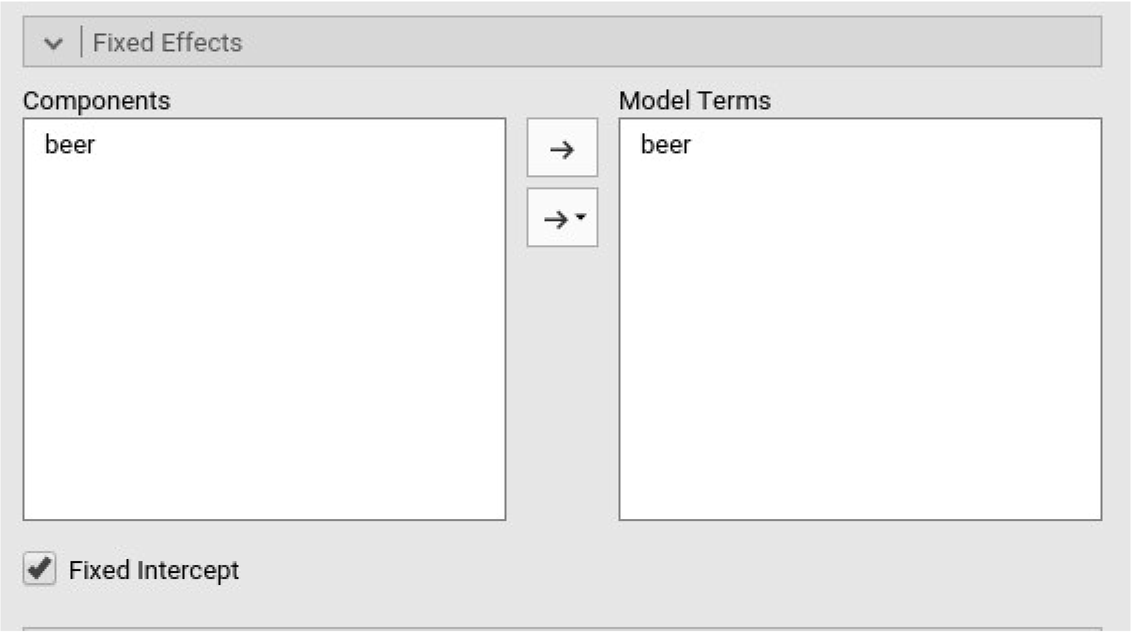
Define the variables role



Jamovi

- In jamovi mixed models can be estimated with the GAMLj module

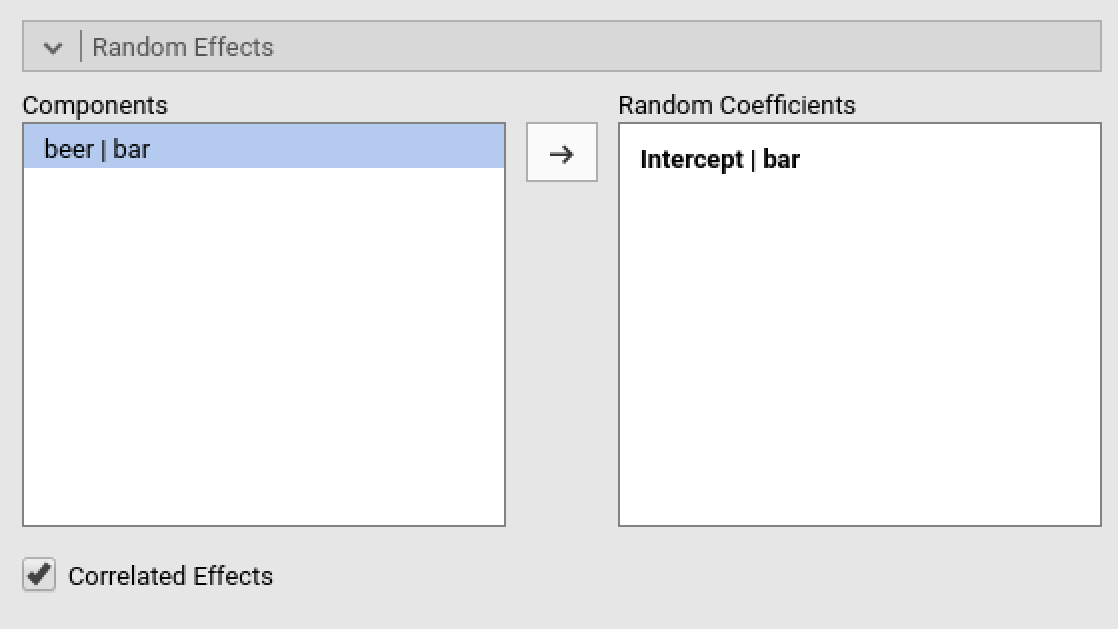
Define the fixed effects



Jamovi

- In jamovi mixed models can be estimated with the GAMLj module

Define the random component



Jamovi

- As soon as you define the random component, you get the results

Mixed Model

Model Info

Info	
Estimate	Linear mixed model fit by REML
Call	smile ~ 1 + (1 | bar) + beer
AIC	811.1613
R-squared Marginal	0.0894
R-squared Conditional	0.8172

R-squared Marginal: How much variance can the **fixed effects alone** explain of the overall variance

R-squared Conditional: How much variance can the **fixed and random** effects together explain of the overall variance

Jamovi

- As soon as you define the random component, you get the results

F-test for the main effect of beer

Fixed Effect ANOVA

	F	Num df	Den df	p
beer	46.0	1	229	< .001

Note. Satterthwaite method for degrees of freedom

Jamovi

- As soon as you define the random component, you get the results

coefficients for the main effect of beer

Fixed Effects Parameter Estimates

Effect	Contrast	Estimate	SE	95% Confidence Interval Lower	95% Confidence Interval Upper	df	t	p
(Intercept)	Intercept	7.778	0.6276	6.548	9.008	13.2	12.39	< .001
beer	beer	0.548	0.0808	0.390	0.706	229.4	6.79	< .001

Jamovi

- As soon as you define the random component, you get the results

Random components

Random Components

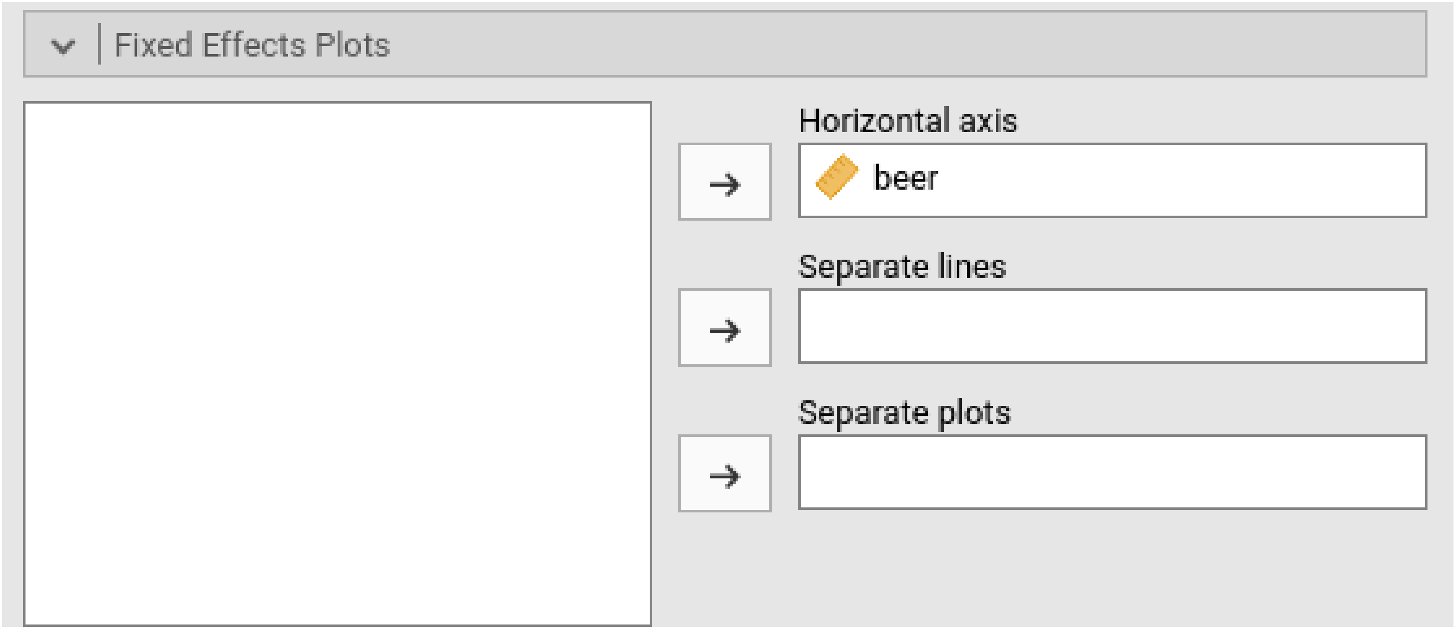
Groups	Name	SD	Variance
bar	(Intercept)	2.40	5.77
Residual		1.20	1.45

Note. Numer of Obs: 234 , groups: bar , 15

Jamovi

- Jamovi can plot up to a 3-way interaction

Plot (here just one line)

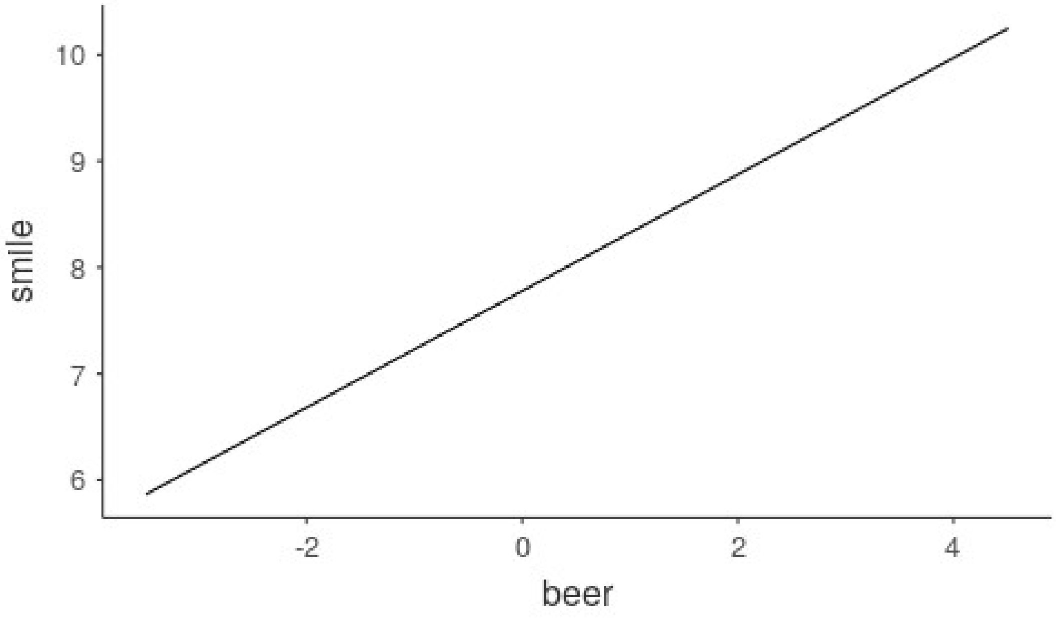


Jamovi

- Jamovi can plot up to a 3-way interaction fixed effects

Plot (here just one line)

Fixed Effects Plots

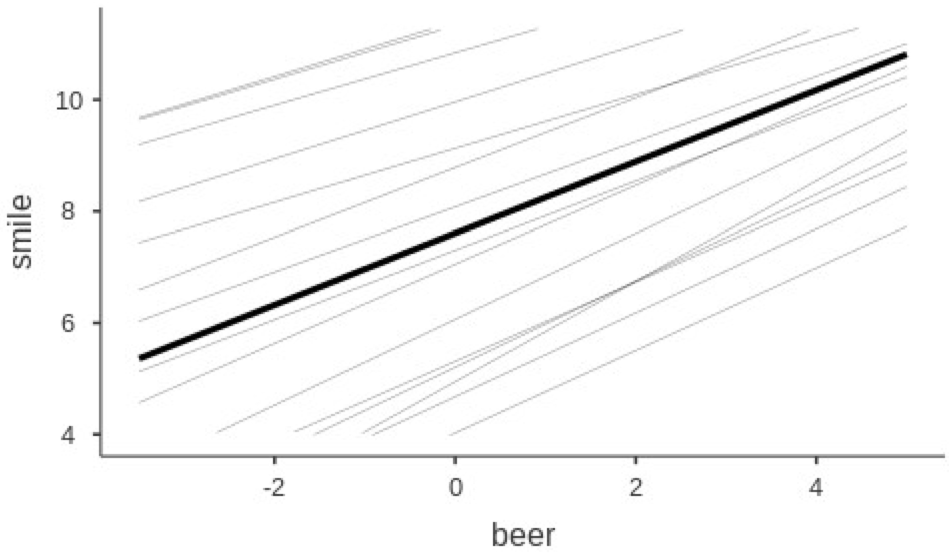


Jamovi

- Jamovi can plot up to a 3-way interaction fixed effects

Random effects

Fixed Effects Plots



Beers at the bar 2

We can now try a model where also the **b** coefficients are allow to vary across clusters

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + b \cdot x_{ij} + e_{ij}$$

- Fixed effects? Intercept and beer effect
- Random effects? Intercepts and b coefficients
- Clusters? bar

Some authors may call this model:
Random-coefficients regression
or
Intercepts- and Slopes-as-outcomes model

SPSS syntax

Now we have all the fixed effects and also the random effects

$$y_{ij} = \bar{a} + a_j + \bar{b} \cdot x_{ij} + b \cdot x_{ij} + e_{ij}$$

```
MIXED smile WITH beer by bar
  /CRITERIA=CIN(95) MXITER(100) MXSTEP(10) SCORING(1) SINGULAR(0.000000000001) HCONVERGE(0,
   ABSOLUTE) LCONVERGE(0, ABSOLUTE) PCONVERGE(0.000001, ABSOLUTE)
  /FIXED=beer| SSTYPE(3)
  /METHOD=REML
 /print solution TESTCOV
  /random intercept beer  | SUBJECT(bar) COVTYPE(un).
```

Fixed effects (intercept is included by default)

Random effects

Cluster variable

SPSS Output

Let's see if the model is how intended

Model Dimension[b]

		Number of Levels	Covariance Structure	Number of Parameters	Subject Variables
Fixed Effects	Intercept	1		1	
	beer	1		1	
Random Effects	Intercept + beer[a]	2	Unstructured	3	bar
Residual				1	
Total		4		6	

a. As of version 11.5, the syntax rules for the RANDOM subcommand have changed. Your command syntax may yield results that differ from those produced by prior versions. If you are using version 11 syntax, please consult the current syntax reference guide for more information.
b. Dependent Variable: smile.

OK!

SPSS Output

We then check the variability of the random effects. If there is variability across bars, it means we were right to model the coefficients as random

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Residual		1.258809	.125848	10.003	.000	1.034814	1.531292
Intercept + beer [subject = bar]	UN (1,1)	9.334205	4.379192	2.131	.033	3.721616	23.411169
	UN (2,1)	-.446262	.434661	-1.027	.305	-1.298181	.405657
	UN (2,2)	.034518	.053446	.646	.518	.001660	.717792

a. Dependent Variable: smile.

Notice that the variance of beer is not significantly different from zero. We do not care

Variance

Variances of random coefficients inform us on the variability of the effects

- Even when they are not significant, we keep them as random
- When variances are exactly zero (and SPSS gives a general warning), effects should be set only as fixed

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Residual		1.258809	.125848	10.003	.000	1.034814	1.531292
Intercept + beer [subject = bar]	UN (1,1)	9.334205	4.379192	2.131	.033	3.721616	23.411169
	UN (2,1)	-.446262	.434661	-1.027	.305	-1.298181	.405657
	UN (2,2)	.034518	.053446	.646	.518	.001660	.717792

a. Dependent Variable: smile.

SPSS Output

If everything is fine, we interpret the fixed effects as in any other GLM (regression)

Estimates of Fixed Effects[a]

Parameter	Estimate	Std. Error	df	t	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Intercept	5.373312	.850910	11.597	6.315	.000	3.512159	7.234465
beer	.641676	.092399	9.336	6.945	.000	.433796	.849555

a. Dependent Variable: smile.

Intercept: On average, for zero beers we expect 5.37 smiles

B coefficient: On average,for each beer more, on average we expect .641 more smiles

Random effect covariance

We noticed that the covariance parameters are 3

Covariance Parameters

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Residual		1.258809	.125848	10.003	.000	1.034814	1.531292
Intercept + beer [subject = bar]	UN (1,1)	9.334205	4.379192	2.131	.033	3.721616	23.411169
	UN (2,1)	-.446262	.434661	-1.027	.305	-1.298181	.405657
	UN (2,2)	.034518	.053446	.646	.518	.001660	.717792

a. Dependent Variable: smile.

Random effect covariance

We noticed that the covariance parameters are 3

Covariance Parameters

Estimates

Parameter		Estimate
Residual		1.258809
Intercept + beer [subject = bar]	UN (1,1)	9.334205
	UN (2,1)	-.446262
	UN (2,2)	.034518

a. Dependent Variable: smile.

	Intercept	B (beer)
Intercept	(1,1) σ_a	
B (beer)	(2,1) σ_{ab}	(2,2) σ_b

Variance of random effects

Random effect covariance

We noticed that the covariance parameters are 3

Covariance Parameters

Estimates

Parameter		Estimate
Residual		1.258809
Intercept + beer [subject = bar]	UN (1,1)	9.334205
	UN (2,1)	-.446262
	UN (2,2)	.034518

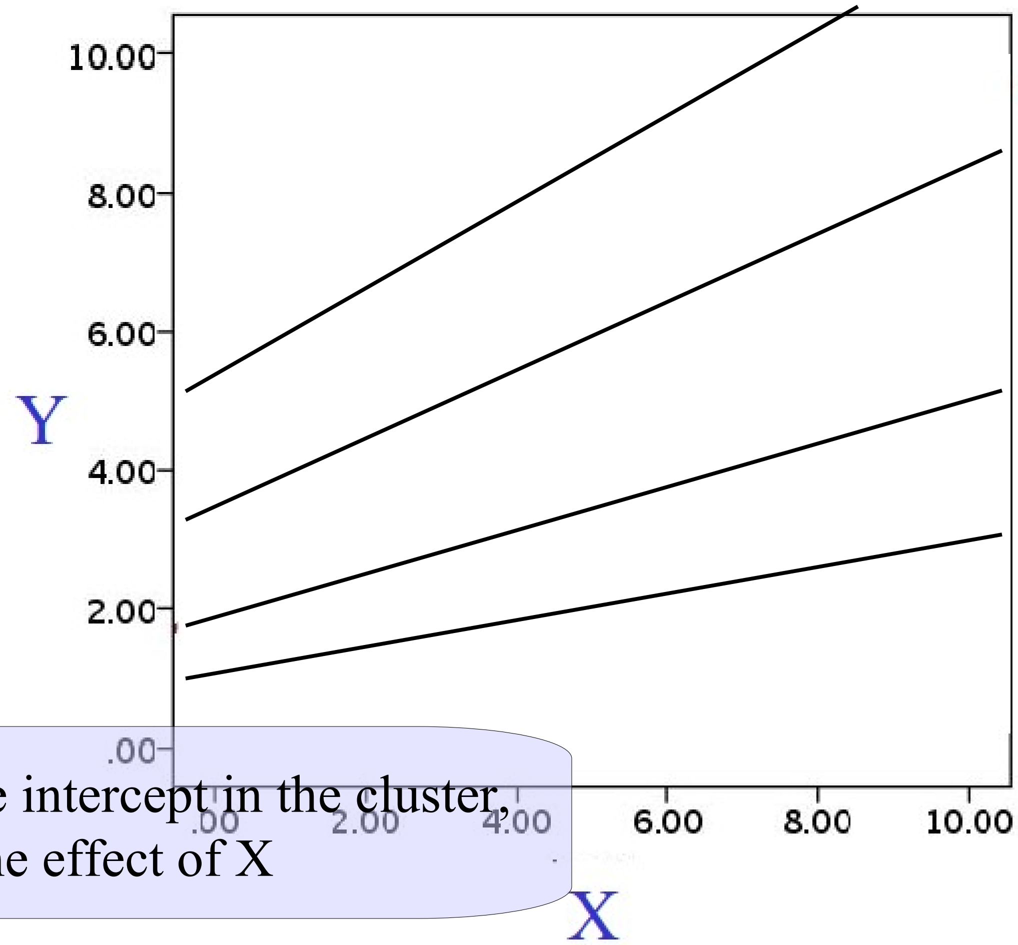
a. Dependent Variable: smile.

	Intercept	B (beer)
Intercept	(1,1) σ_a	
B (beer)	(2,1) σ_{ab}	(2,2) σ_b

Covariance between random coefficients

Covariance

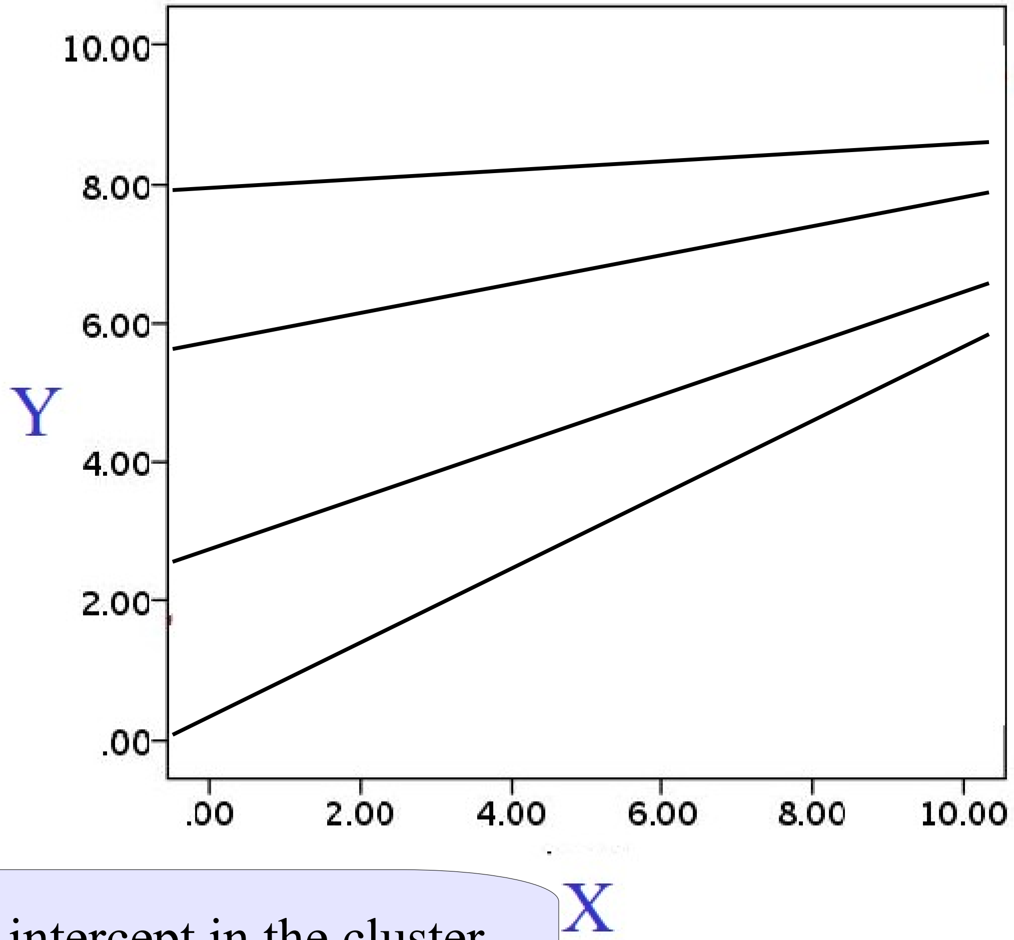
Example of positive covariance between **a** and **b**



The higher the intercept in the cluster, the stronger the effect of X

Covariance

Example of negative covariance between **a** and **b**

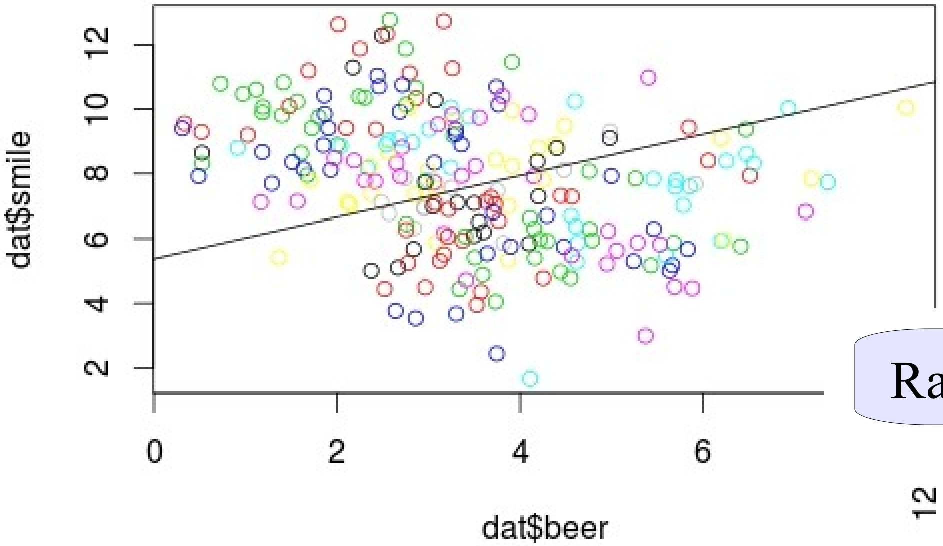


The higher the intercept in the cluster, the weaker the effect of X

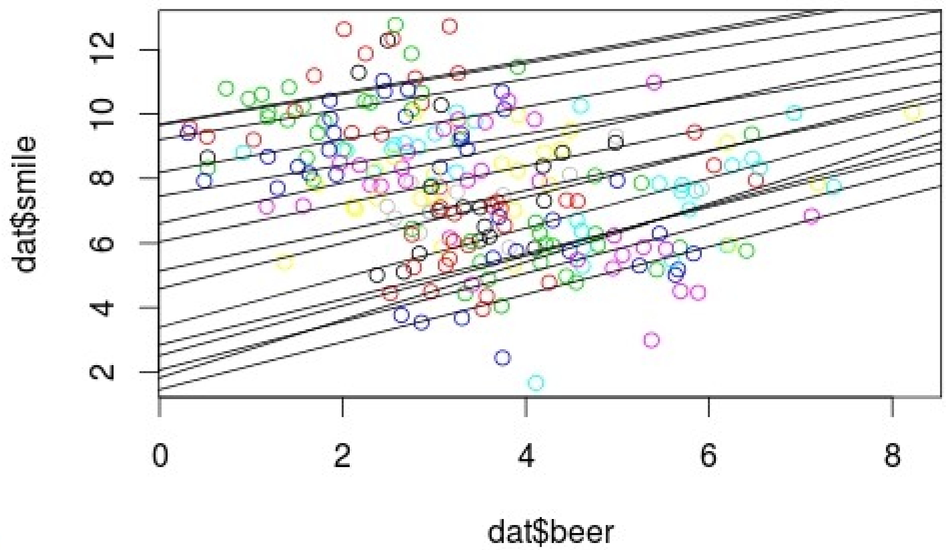
Plots

Plots

Fixed effects



Random effects



Mixed Linear Models

- With the mixed model one can take into the account dependency among measures (within clusters) almost in any situation
- It allows applying the GLM logic to a broader range of designs
- Interactions with any kind of variable
- Efficient handling of missing values
- Continuous repeated measures variables
- Hierarchical organization of the data

Repeated Measures ANOVA as a linear mixed model

Next Meeting

The End