Chapter 4 - Emission of radiation from plasmas

1 Black body radiation spectrum

A black body can be made by a cavity with a small entrance hole so that all the impinging radiation is trapped in the cavity.

a) Consider a cubic cavity of side L and the wave equation

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{1}$$

where $\phi(x, y, z, t)$ is a component of the electric field in the cavity and c is the speed of light. By searching for separable solutions of the type

$$\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \tag{2}$$

and by imposing that ϕ vanishes at any side of the cubic cavity, show that the eigenfunctions of equation 1 are

$$\phi(x, y, z, t) = A \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t + \alpha) \tag{3}$$

where $k_i = \frac{\pi}{L} n_i$, $i = x, y, z$, $n_i = 1, 2, 3 \ldots$, $\omega = kc$, $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ and α is an initial phase.
Show in particular that photons in the cavity carry the energy

$$E = \frac{hc}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2} \tag{4}$$

where h in the Planck's constant.

b) Let's denote with U the total energy in the cavity. Then

$$U = \sum_{all E} E \times \mathscr{P}(E) \times \mathscr{N}(E) \tag{5}$$

where $\mathscr{P}(E)$ is the probability that photons have the energy E and $\mathscr{N}(E)$ is the number of states at the energy E.
Since the Bose-Einstein statistics applies,

$$\mathscr{P}(E) = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{6}$$

Here ν is the photon frequency, k_B is the Boltzmann constant and T indicates temperature.
Concerning the density of states, show that[1]

$$\mathcal{N}(E) = \left(\frac{2L}{c}\right)^3 \pi\nu^2 d\nu \tag{7}$$

By denoting with u_ν the *energy density* in the cavity, show that

$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{8}$$

c) We now want to evaluate the intensity of the radiation emitted by the cavity per unit frequency, i.e. $\frac{dI}{d\nu}$.
The black body is in thermal equilibrium, i.e. the intensity of the radiation that is absorbed by the black body must also be emitted at the same rate and with the same spectrum.
If the black body is immersed in an isotropic flux of photons with an energy density u_ν, show that

$$\frac{dI}{d\nu} = \frac{c}{4}u_\nu \tag{9}$$

i.e. that

$$\frac{dI}{d\nu} = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{10}$$

2 Electron bremsstrahlung radiation spectrum: a simplified approach

An electron with a velocity v has an impact parameter b from a ion having a charge Ze. The bremsstrahlung interaction time is $\tau \approx 2b/v$ so that the dominant frequency of the radiation emitted by the electron is $\nu = 1/\tau = v/(2b)$. Considering that the number of collisions per second at an impact parameter b is $2\pi n_i v b db$, where n_i is the ion density, show that

$$2\pi n_i v b db = \frac{\pi n_i v^3}{2\nu^3} d\nu \tag{11}$$

Moreover, by making using of the formula for the total power $P(t)$ radiated by a charged particle with an acceleration a, show that the energy lost by the electron during each interaction is

[1] *Hints:*

- Note that states in a spherical crust of thickness dn in the (n_x, n_y, n_z) space have all the same energy.
- n_x, n_y, n_z are *positive* integers and there are *two* polarization states of the photon at a given energy.

$$\Delta E \approx P(t)\tau = \frac{\mu_0 Z^2 e^6}{48\pi^3 c\epsilon_0^2 m_e^2} \frac{1}{b^3 v} \tag{12}$$

where μ_0 and ϵ_0 are the magnetic permeability and electric permittivity, respectively.

Multiply equations 11 and 12 to show that the energy lost per unit time is

$$\frac{dE}{dt} = \frac{\mu_0 Z^2 e^6 n_i}{12\pi^2 c\epsilon_0^2 m_e^2} \frac{1}{v} d\nu \tag{13}$$

Finally, assuming a Maxwellian electron distribution and noting that $v > \sqrt{\frac{2h\nu}{m_e}}$, integrate equation 13 over all velocities to show that the power spectrum of the electron bremsstrahlung radiation is approximately

$$W(\nu) \approx \frac{\mu_0 Z^2 e^6 n_i n_e}{6\pi^{5/2} c\epsilon_0^2 m_e^2} \left(\frac{m_e}{2k_B T_e}\right)^{1/2} \exp\left(-\frac{h\nu}{k_B T_e}\right) \tag{14}$$