Chapter 6 - Introduction to thermonuclear fusion

1 Tunnel effect and fusion cross section

The effective potential seen by two particles 1 and 2 for a fusion reaction in the centre of mass frame can be represented as in figure 1. The two particles have the charge $+Z_1 e$ and $+Z_2 e$, respectively, a reduced mass μ and a relative kinetic energy ϵ. The nucleus has a radius $r_n = r_{n1} + r_{n2}$ and r_{tp} indicates the classical turning point. The potential well of the nucleus has a depth $-U_0$ and the Coulomb barrier has a maximum height of V_b.
Classically, a particle with relative kinetic energy ϵ cannot overcome the Coloumb barrier if $\epsilon < V_b$. Quantum mechanics however predicts that the particle can overcome the barrier even when $\epsilon < V_b$ due to tunneling and can thus undergo a fusion reaction also in this case.

The solution of the Schrödinger equation for this problem shows that the effective potential seen by the wave component that carries the angular momentum $\hbar\sqrt{l(l+1)}$ is

$$V_{eff,l}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2} \tag{1}$$

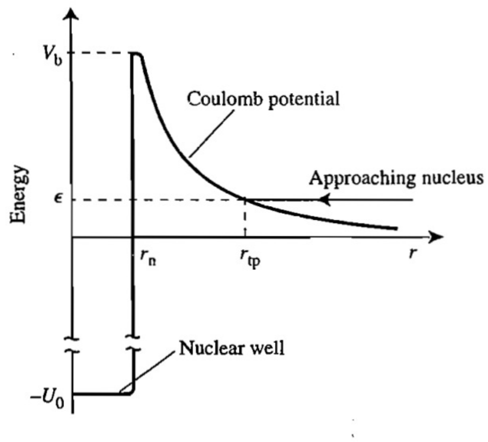

Figure 1: Effective potential in the centre of mass frame seen by a charged particle with relative kinetic energy ϵ undergoing a fusion reaction.

where $V(r)$ is the potential of figure 1. The angular momentum components of the wave that have $l \neq 0$ experience a progressively increased potential barrier due to the centrifugal term $\frac{\hbar^2 l(l+1)}{2\mu r^2}$ and have a smaller probability to undergo a fusion reaction. We can thus assume that, to a first extent, only the $l = 0$ component contributes to the reaction, so that $V_{eff,l}(r) = V(r)$; in particular, when $r > r_n$, $V(r)$ is the repulsive Coulomb potential between two charges $+Z_1 e$ and $+Z_2 e$.
Under these conditions, the WKB theory of quantum mechanics predicts that the tunneling probability P is

$$P \approx \exp\left(-\frac{2}{\hbar}\int_{r_n}^{r_{tp}} \sqrt{2\mu\left(V(r)-\epsilon\right)}dr\right) \tag{2}$$

Evaluate P and show that, if $r_{tp} >> r_n$

$$P \approx \exp\left(-\sqrt{\frac{\epsilon_G}{\epsilon}}\right) \tag{3}$$

$\epsilon_G = (\pi\alpha_f Z_1 Z_2)^2 2mc^2$ is the "Gamow energy" and $\alpha_f = \frac{e^2}{4\pi\epsilon_0\hbar c}$ is the fine structure constant [1].

2 Some estimates for fusion power plants

A fusion power plant operates at the temperature T=15 keV, when the t(d, n)α reactivity is $\approx$ $2.7 \cdot 10^{-22}$ m^3/s so to produce 1 GW of thermal fusion power. If the plasma has an electron density n = 10^{20} m^{-3}

a) How many reactions per second need to occur?

b) What is the plasma volume that allows obtaining this amount of reactions per second?

c) Considering that deuterium has an isotopic abundance of 0.015%, how many litres of water have to be used to produce all the deuterium in the power plant?

d) What is the fraction per second of deuterium that is burnt due to the fusion reactions?

[1] The following recursive formula can be useful when evaluating the integral for P. If $J_n = \int \frac{dt}{(1+t^2)^n}$, then

$$J_{n+1} = \frac{1}{2n}\frac{t}{(1+t^2)^n} + \frac{2n-1}{2n}J_n \tag{4}$$