Graph Theory and Algorithms PhD. course
Graph hyperbolicity

Mauricio Soto Gomez

DIG, Politecnico di Milano

May 4, 2021

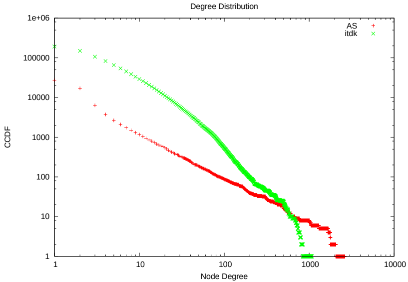
“Empirical” Networks

Networks from different domains exhibit common properties:

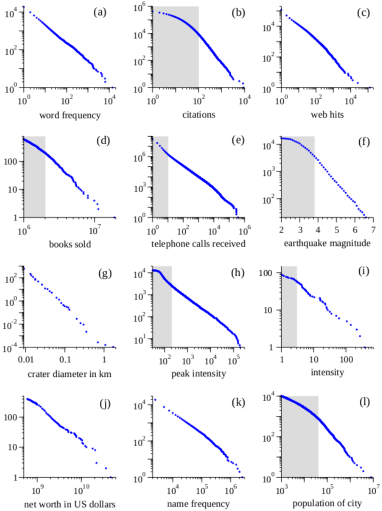
- sparse
- small diameter
- locally dense
- scale free degree distribution:
  proportion of nodes of degree k: $P(k) \sim k^{-\gamma}, 2 < \gamma < 3$

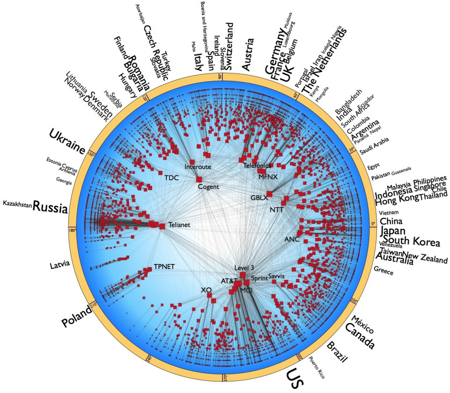
For example:

- Scientific citation
- Internet interconnection
- Social networks



Power Law distributions





a

[a] https://www.caida.org

Tree-like structure

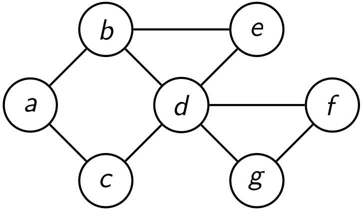
- Simple
- Decomposable → recursive
- Efficient algorithms:
  - E.g. Longest path in $O(|E|)$ (two BFS's)
  - ...and not only:
    https://www.graphclasses.org/classes/gc_342.html

So how to measure tree-like graphs

- **Treewidth** → structure
- **Hyperbolicity** → metric

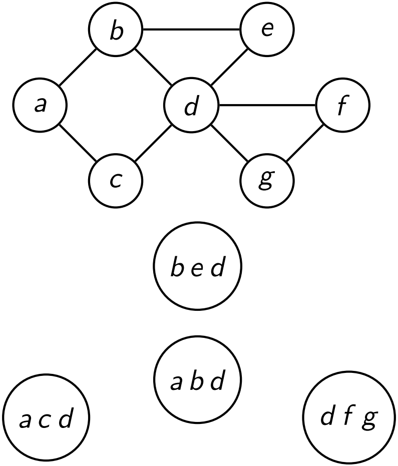
Treewidth

Tree-decomposition [Robertson & Seymour '84]



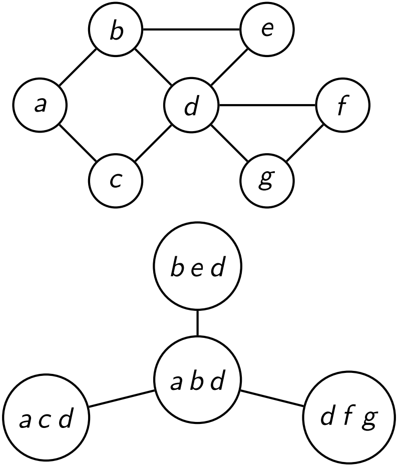
Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$



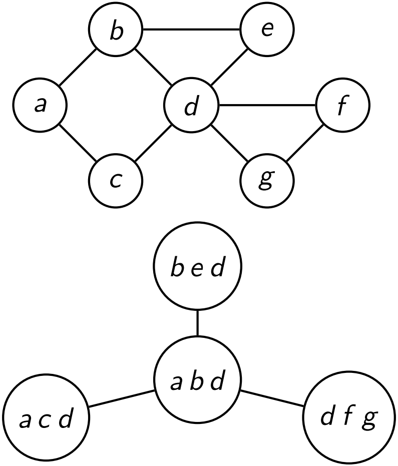
Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$
- Tree $T = (I, F)$



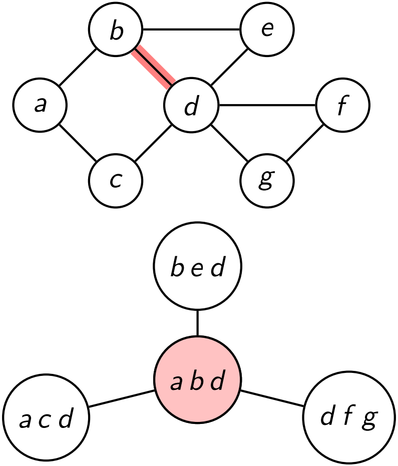
Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$
- Tree $T = (I, F)$
  1. every vertex is in a bag.



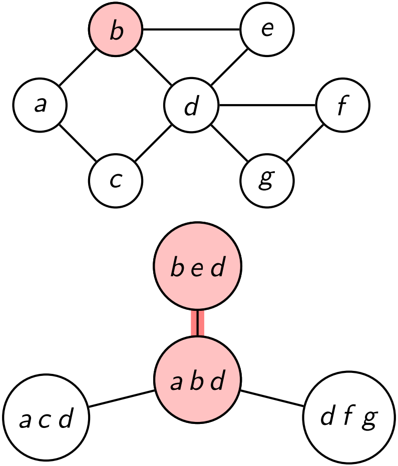
Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$
- Tree $T = (I, F)$
  1. every vertex is in a bag.
  2. endpoints of any edge are in a bag.



Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$
- Tree $T = (I, F)$
  1. every vertex is in a bag.
  2. endpoints of any edge are in a bag.
  3. bags with a same vertex induce a connected sub-tree.



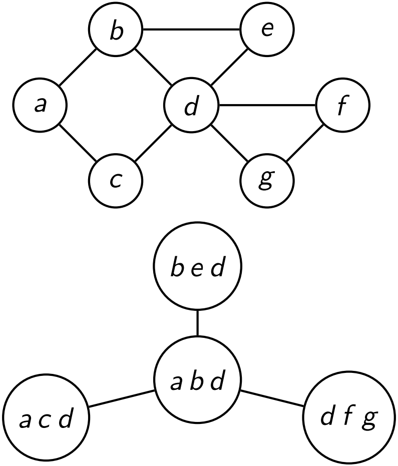
Tree-decomposition [Robertson & Seymour '84]

- Bags $\{X_i \subseteq V, i \in I\}$
- Tree $T = (I, F)$
  1. every vertex is in a bag.
  2. endpoints of any edge are in a bag.
  3. bags with a same vertex induce a connected sub-tree.

Width: $\max_i |X_i| - 1$

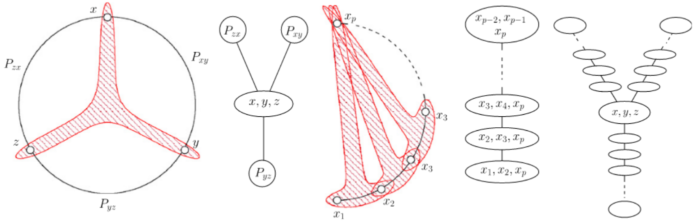
Treewidth

$tw(G)$ the smallest width over all tree-decompositions.



Treewidth : Examples

	Treewidth
Tree	1
Clique	n-1
Grid $n \times m$	$\min\{n, m\}$
Cycle	2



[1]

Assignment: find the tree-decomposition of a grid of width $\min\{n, m\}$.

[1] Dourisboure, Yon & Gavoille, Cyril. (2007). Tree-Decompositions with Bags of Small Diameter. Discrete Mathematics. 307. 2008-2029.

Treewidth : Algorithmic Motivation

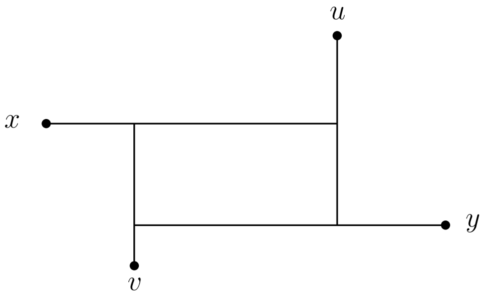
Good. Many NP-hard problems have polynomial time algorithms for bounded treewidth graphs: Hamiltonian Circuit, Independent Set, Vertex Cover...

Bad. Determining the tree-width of a graph is NP-hard [S. Arnborg, D.G. Corneil, A. Proskurowski '87]

Hyperbolicity

Hyperbolicity: Definition

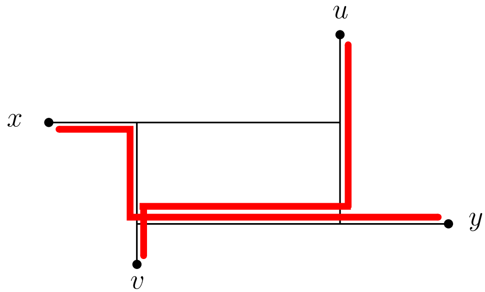
[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

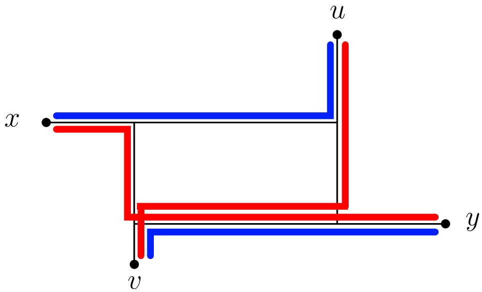
$xy + uv$



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

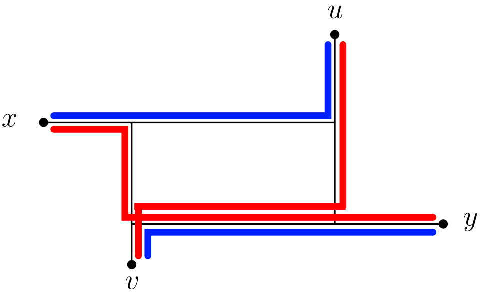
$xy + uv \geqslant xu + yv$



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

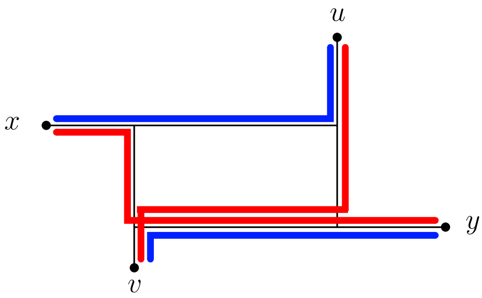
$xy + uv \geqslant xu + yv \geqslant xv + yu$



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

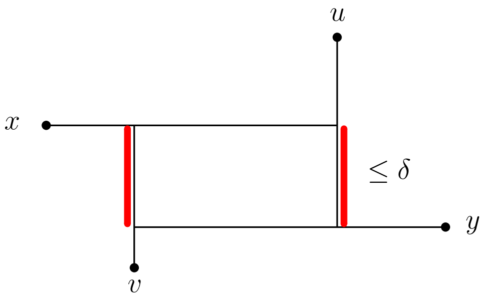
$xy + uv \geqslant xu + yv \geqslant xv + yu$
then
$xy + uv$-$(xu + yv) \leqslant 2\delta$



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

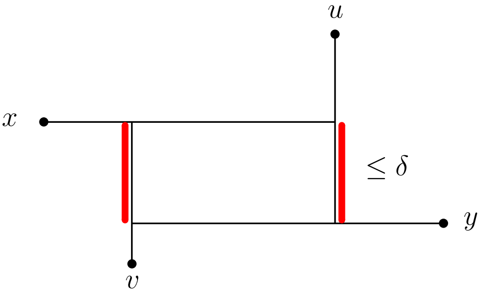
$xy + uv \geqslant xu + yv \geqslant xv + yu$
then
$xy + uv$-$(xu + yv) \leqslant 2\delta$



Hyperbolicity: Definition

[Gromov '87]: (X, d) metric space is δ-*hyperbolic* if:
$\forall u, v, x, y \in X$ with

$xy + uv \geqslant xu + yv \geqslant xv + yu$
then
$xy + uv$-$(xu + yv) \leqslant 2\delta$



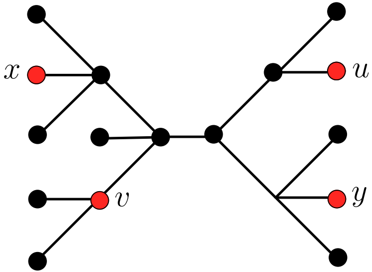
Complexity: $O(n^4)$
polynomial but long in practice!

Hyperbolicity : Examples

	Hyperbolicity
Tree	

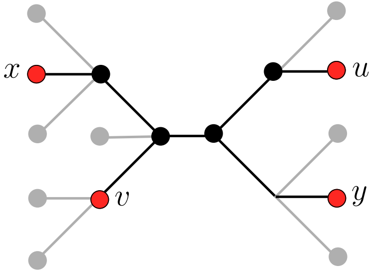
Hyperbolicity : Examples

	Hyperbolicity
Tree	



Hyperbolicity : Examples

	Hyperbolicity
Tree	0



Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	

Hyperbolicity : Examples

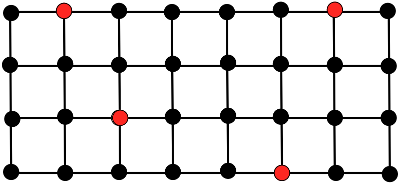
	Hyperbolicity
Tree	0
Clique	0

Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0

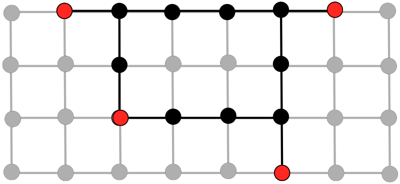
Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0
Grid $n \times m$	



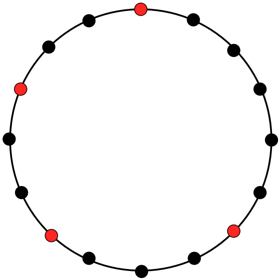
Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0
Grid $n \times m$	$\min\{n, m\}$



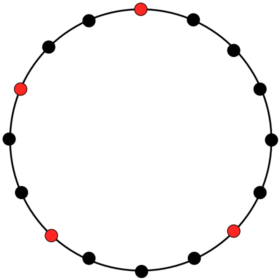
Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0
Grid $n \times m$	$\min\{n, m\}$
Cycle	$n/4$



Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0
Grid $n \times m$	$\min\{n, m\}$
Cycle	$n/4$



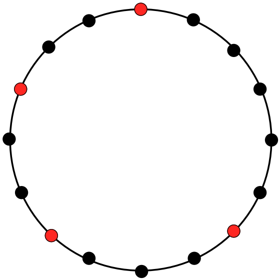
[Wu et Zhang '11]

$$\delta(G) \leqslant \frac{\text{chordality}(G)}{4}$$

.

Hyperbolicity : Examples

	Hyperbolicity
Tree	0
Clique	0
[Brinkmann et Koolen, '01]	
Block graph	0
Grid $n \times m$	$\min\{n, m\}$
Cycle	$n/4$



[Wu et Zhang '11]

$$\delta(G) \leqslant \frac{\text{chordality}(G)}{4}$$

.

Assignment:** Prove that chordal graphs are 1-hyperbolic.

Treewidth vs Hyperbolicity

	Hyperbolicity	Treewidth
Tree	0	1
Clique	0	n-1
Grid $n \times m$	$\min\{n, m\}$	$\min\{n, m\}$
Cycle	n/4	2

Tree metric approximation

Theorem [Gromov 87], [Ghys and Harpe 90]

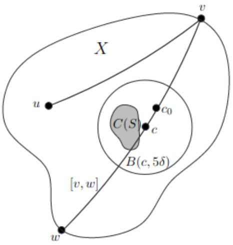
Given a finite δ-hyperbolique metric space (X, d) with $|X| = n$ and a root $s \in X$, there exists a weighted tree T and a function $\phi : X \to T$ such that

- $\forall x \in X$, $d(s, x) = d_T(\phi(s), \phi(x))$
- $\forall x, y \in X$, $d(x, y) - 2\delta \log_2(n) \leqslant d_T(\phi(x), \phi(y)) \leqslant d(x, y)$

The construction of function ϕ and tree T can be done in $O(n^2)$.

Hyperbolicity and Longest path

[Chepoi et al '08]: it suffices (almost) two BFS searches.



Lemma

Let (X, d) be a δ-hyperbolic space. If $d(v, u) \geqslant \max\{d(y, u), d(x, u)\}$, then $d(x, y) \leqslant \max\{d(v, x), d(v, y)\} + 2\delta$.

Theorem

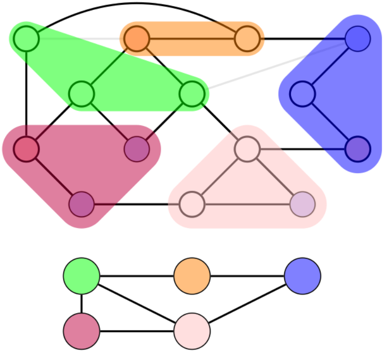
$d(v, w) \geqslant diam(G) - 2\delta$.

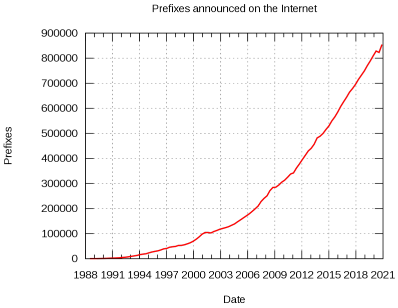
Internet routing

```
traceroute biblio.math.jussieu.fr
traceroute to biblio.math.jussieu.fr (134.157.100.3), 30 hops max, 40 byte packets
 1  gw.dim.uchile.cl (146.83.7.1)  0.278 ms  0.287 ms  0.274 ms
 2  172.17.32.1 (172.17.32.1)  1.348 ms  1.710 ms  1.962 ms
 3  cisco-dti.cec.uchile.cl (200.9.98.130)  1.559 ms  1.796 ms  2.017 ms
 4  172.16.38.1 (172.16.38.1)  3.989 ms  4.036 ms  4.025 ms
 5  200.89.75.37 (200.89.75.37)  4.005 ms  4.092 ms  3.981 ms
 6  rci2-uchile.reuna.cl (146.83.242.25)  4.142 ms  4.039 ms  4.005 ms
 7  reuna-cl-sant.core.redclara.net (200.0.204.141)  4.101 ms  4.481 ms  4.712 ms
 8  saopaulo-santiago.core.redclara.net (200.0.204.38)  53.259 ms  53.241 ms  53.230 ms
 9  RedCLARA.rt1.mad.es.geant2.net (62.40.124.137)  291.246 ms  291.467 ms  291.329 ms
10  so-4-1-0.rt1.par.fr.geant2.net (62.40.112.118)  307.726 ms  308.380 ms  308.364 ms
11  renater-gw.rt1.par.fr.geant2.net (62.40.124.70)  311.907 ms  311.896 ms  311.884 ms
12  te1-1-paris1-rtr-021.noc.renater.fr (193.51.189.38)  309.027 ms  309.008 ms  308.746 ms
13  * * *
14 rap-vl165-te3-2-jussieu-rtr-021.noc.renater.fr (193.51.181.101)  307.888 ms 324.131 ms
322.731 ms
15  site-6.01-jussieu.rap.prd.fr (195.221.127.182)  307.927 ms  308.137 ms  308.487 ms
16  r-jussieu2.reseau.jussieu.fr (134.157.254.14)  308.512 ms  308.438 ms  308.426 ms
17  134.157.100.3 (134.157.100.3)  308.584 ms  307.828 ms  307.464 ms
```

```
BGP table version is 334348919, local router ID is 128.223.51.108
Status codes: s suppressed, d damped, h history, * valid, > best, i - internal,
              r RIB-failure, S Stale
Origin codes: i - IGP, e - EGP, ? - incomplete

   Network          Next Hop            Metric LocPrf Weight Path
r> 0.0.0.0          95.140.80.253                          0 31500 29632 13307 i
*  2.0.0.0/16       74.40.7.35               0             0 5650 6939 30132 12654 i
*                   72.36.126.8                            0 40387 559 30132 12654 i
*                   64.57.28.241           904             0 11537 20965 559 30132 12654 i
*>                  195.215.109.252                        0 3292 12654 i
*                   218.189.6.129                          0 9304 6939 30132 12654 i
*                   207.246.129.1                          0 11608 6939 30132 12654 i
*                   200.160.127.238                        0 27664 16735 12956 12654 i
*                   195.215.109.254      65469             0 3292 12654 i
*                   216.218.252.164                        0 6939 30132 12654 i
...
*  4.79.22.0/23     74.40.7.35               0             0 5650 3356 i
*                   200.160.127.238                        0 27664 16735 12956 3356 i
*                   95.140.80.253                          0 31500 3267 3216 1273 3356 i
*>                  134.222.87.4                           0 286 3356 i
*  4.79.181.0/24    74.40.7.35               0             0 5650 10310 14780 i
*                   72.36.126.8                            0 40387 11537 10310 14780 i
*                   207.246.129.1                          0 11608 10310 14780 i
*                   195.215.109.252                        0 3292 10310 14780 i
*                   207.246.129.2                          0 11608 10310 14780 i
*                   200.160.127.238                        0 27664 16735 12956 10310 14780 i
*                   216.218.252.164                        0 6939 10310 14780 i
*                   195.215.109.254      75499             0 3292 10310 14780 i
*                   202.147.0.142        10003             0 10026 10310 14780 i
*>                  89.149.178.10           10             0 3257 12180 12180 i
...
```





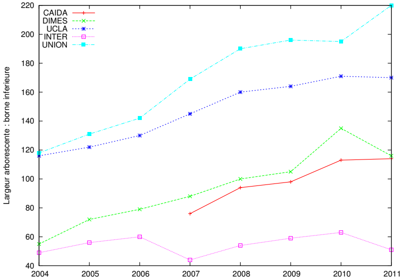
Routing on tree-like graphs

1. [Widmayer, Neyer et Eidenbenz '99] routing tables of size $O(tw(G)\log^2(|V|))$.
2. [Dourisboure '02] routing tables of size $O(k\log(|V|))$ for chordal graphs (k max clique size).
3. [Chepoi & *al.* '10] routing tables of size $O(\delta\log|V|)$.

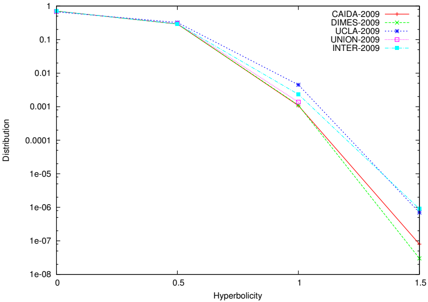
Internet graphs (2009)

		$\lvert V \rvert$	Avg.degree	Hyperbolicity
AS	CAIDA	29797	5,31	1.5
	UCLA	37450	6,65	1.5

tw for AS graph



hyperbolicity for AS quadruplets

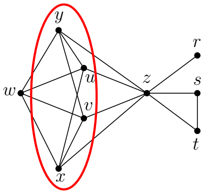


Hyperbolicity and graph decomposition

Modular Decomposition

Definition ([Gallai 67])

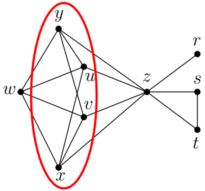
- A *module* of a graph is a subset $M \subseteq V(G)$ indistinguishable from exterior vertices: $\forall x \in V \setminus M$, either $M \subseteq N(x)$, or $M \cap N(x) = \emptyset$.



Modular Decomposition

Definition ([Gallai 67])

- A *module* of a graph is a subset $M \subseteq V(G)$ indistinguishable from exterior vertices: $\forall x \in V \setminus M$, either $M \subseteq N(x)$, or $M \cap N(x) = \emptyset$.

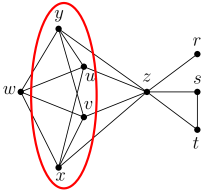


- A module M is *strong* if there is no another module M' such that $M \cap M'$, $M \setminus M'$ and $M' \setminus M$ are not empty.

Modular Decomposition

Definition ([Gallai 67])

- A *module* of a graph is a subset $M \subseteq V(G)$ indistinguishable from exterior vertices: $\forall x \in V \setminus M$, either $M \subseteq N(x)$, or $M \cap N(x) = \emptyset$.



- A module M is *strong* if there is no another module M' such that $M \cap M'$, $M \setminus M'$ and $M' \setminus M$ are not empty.
- A graph is *prime* if it does not contain a non-trivial module.

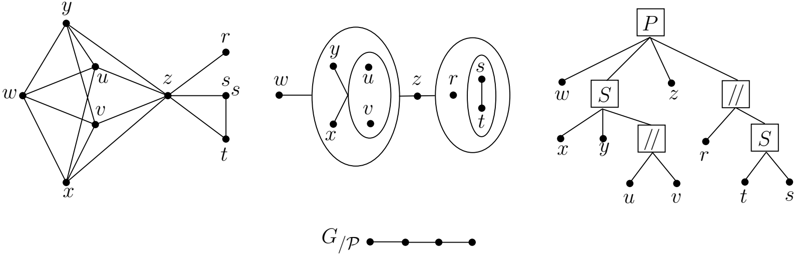
Theorem of Modular Decomposition

[Chein, Habib Maurer 81, Gallai 67]

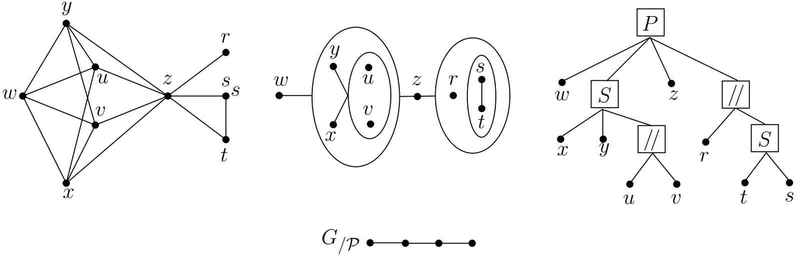
For every graph G,

1. G is non-connected (parallel case). Strong maximal modules are their connected components.
2. $\bar{G}$ is non-connected (series case). Strong maximal modules are the connected components of $\bar{G}$.
3. G and $\bar{G}$ are connected (prime case). Every maximal module of G is strong.

Hyperbolicity and Modular Decomposition

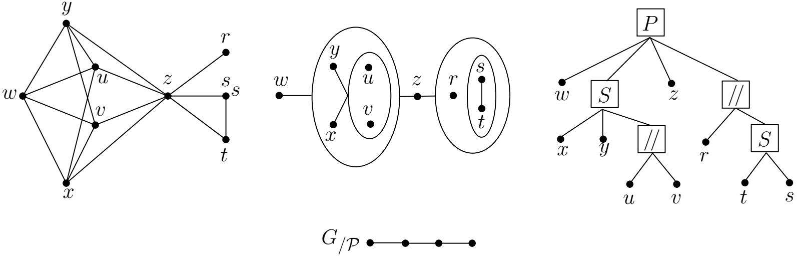


Hyperbolicity and Modular Decomposition



[Cournier et Habib 94] : Modular decomposition can be computed in $O(|V| + |E|)$.

Hyperbolicity and Modular Decomposition



[Cournier et Habib 94] : Modular decomposition can be computed in $O(|V| + |E|)$.

Lemma

Let $M_x, M_y \in \mathcal{P}$ the modules such that $x \in M_x$ and $y \in M_y$. Then $d_G(x, y) = d_{G_{/\mathcal{P}}}(M_x, M_y)$ if $M_x \neq M_y$, or $d_G(x, y) \leqslant 2$ otherwise.

Theorem

$\delta(G) \leqslant \max\left\{\delta(G_{/\mathcal{P}}), 1\right\}$

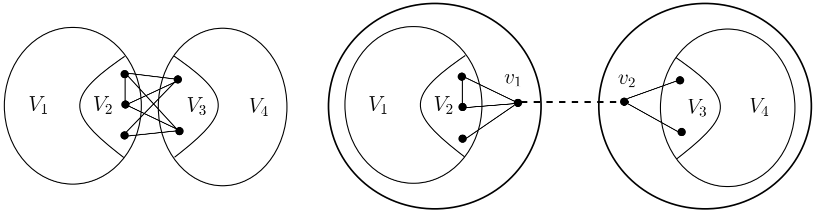
Corollary

Cographs (P_4-free) are 1-hyperbolique.

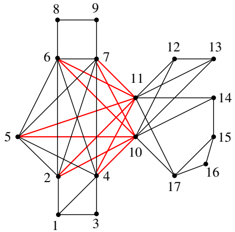
Split-decomposition

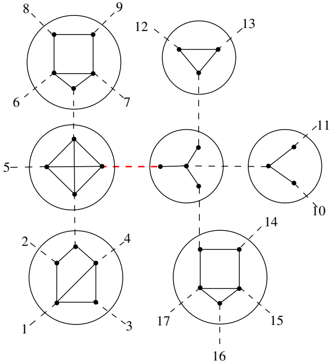
Definition [Cunningham 82]

A *split* of G is a partition X_1, X_2 such that
$\forall x \in X_1 \cap \mathcal{N}(X_2), \forall y \in X_2 \cap \mathcal{N}(X_1) \Rightarrow (x, y) \in E(G)$.



Split Decomposition Tree



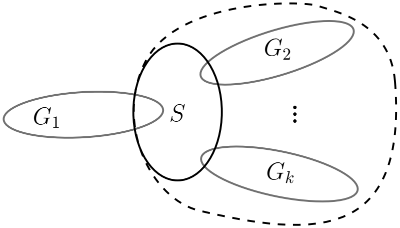


[Dahlhaus 00, Charbit, de Montgolfier et Raffinot 09] : The split decomposition tree can be computed in $O(|V| + |E|)$.

Theorem

$$\delta(G) \leqslant \max\{\{\delta(H), H \subseteq G \text{ and } H \text{ is prime for split}\}, 1\}$$

Hyperbolicity and separators



Proposition

$\max_i \{\delta(G_i \cup S)\} \leqslant \delta(G)$

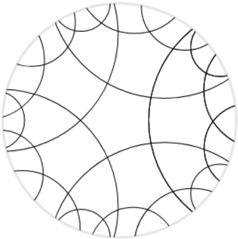
$$\delta(G) \leqslant \max\{\max_i\{\delta(G_i \cup S)\} + \mathrm{diam}_G(S)/2, \mathrm{diam}_G(S)\}$$

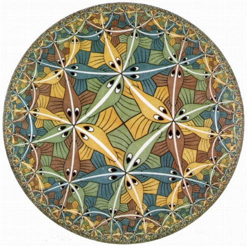
Hyperbolic embeddings

The Poincarè disk model

- The domain is the interior of the disk $D = \{x \in \mathbb{R}^2 : \|x\| < 1\}$,
- endowed with the Poincaré metric:

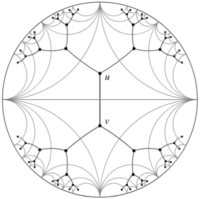
$$d_P(x, y) = \operatorname{arccosh}\left(1 + \frac{2\,\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}\right).$$

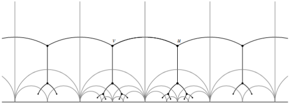


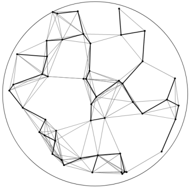


The Poincarè disk model

Trees can be embedded just in dimension two, instead of $\log(|leaves|)$ as in the Euclidean space [Linial, London & Rabinovich '94]







Routing can be done by assigning a virtual coordinate in the hyperbolic space and locally choosing a neighbor closer to the destination.

Thank you!